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ELEMENTS

OF

GEOMETRY,

BY

A. M. LEGENDRE,

MEMBER OF THE INSTITUTE AND THE LEGION OF HONOUR, OF THE ROYAL SOCIETY OF LONDON, &c.

Translated from the French

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ADVERTISEMENT.

The work of M. Legendre, of which the following is a translation, is thought to unite the advantages of modern discoveries and improvements with the strictness of the ancient method. It has now been in use for a considerable number of years, and its character is sufficiently established. It is generally considered as the most complete and extensive treatise on the elements of geometry which has yet appeared. It has been adopted as the basis of the article on geometry in the fourth edition of the Encyclopædia Brittanica, lately published, and in the Edinburgh Encyclopædia, edited by Dr. Brewster.

In the original the several parts are called books, and the propositions of each book are numbered after the manner of Euclid. It was thought more convenient for purposes of reference to number definitions, propositions, corollaries, &c., in one continued series. Moreover the work is considered as divided into two parts, one treating of plane figures and the other of solids; and the subdivisions of each part are denominated sections.

The translator has omitted a number of propositions on spherical isoperimetrical figures terminating the third section of the second part, an appendix to the second and third sections of this part on regular polyedrons, and most of the notes. These are printed in a smaller character in the original to denote that they are less useful, or that they require more attention than other parts of the work. Also the articles numbered in the translation from 229 to 235 inclusive, and from 295 to 312 inclusive, and article 348, are distinguished by the author in the

same manner; and may be passed over or not in the first reading at the discretion of the teacher.

As the reader is supposed to be acquainted with algebraical signs and the theory of proportions, a brief explanation of these, taken chiefly from Lacroix's geometry, and forming properly a supplement to his arithmetic, is prefixed to the work under the title of an introduction.

The translation is from the 11th edition, printed at Paris in 1817.

Cambridge, April, 1819.

PREFACE.

THE method of the ancients is very generally regarded as the most satisfactory and the most proper for representing geometrical truths. It not only accustoms the student to great strictness in reasoning, which is a precious advantage, but it offers at the same time a discipline of peculiar kind, distinct from that of analysis, and which in important mathematical researches may afford great assistance towards discovering the most simple and elegant solutions.

I have thought it proper therefore to adopt in this work the same method which we find in the writing of Euclid and Archimedes; but in following nearly these illustrious models I have endeavoured to improve certain points of the elements which they left imperfect, and especially the theory of solids, which has hitherto been the most neglected.

The definition of a straight line being the most important of the elements, I have wished to be able to give to it all the exactness and precision of which it is susceptible. Perhaps I might have attained this object by calling a straight line that which can have only one position between two given points. For, from this essential property we can deduce all the other properties of a straight line, and particularly that of its being the shortest between two given points. But in order to this it would have been necessary to enter into subtile discussions, and to distinguish, in the course of several propositions, the straight line drawn between two points from the shortest line which measures the distance of these same points. I have preferred, in order not to render the introduction to geometry too difficult, to sacrifice something of the exactness at which I aimed. Accordingly I shall call a *straight line* that which is the shortest between two points, and I shall suppose that there can be only one between the same points. It is upon this principle, considered at the same time as a definition and an axiom, that I have endeavoured to establish the entire edifice of the elements.

It is necessary to the understanding of this work that the reader should have a knowledge of the theory of proportions, which is explained in common treatises either of arithmetic or algebra; he is supposed also to be acquainted with the first rules of algebra; such as the addition and subtraction of quantities, and the most simple operations belonging to equations of the first degree. The ancients, who had not a knowledge of algebra, supplied the want of it by reasoning and by the use of proportions, which they managed with great dexterity. As for us, who have this instrument in addition to what they possessed, we should do wrong not to make use of it, if any new facilities are to be derived from it. I have accordingly not hesitated to employ the signs and operations of algebra, when I have thought it necessary, but I have guarded against involving in difficult operations what ought by its nature to be simple; and all the use I have made of algebra in these elements consists, as I have already said, in a few very simple rules which may be understood almost without suspecting that they belong to algebra.

Besides, it has appeared to me that, if the study of geometry ought to be preceded by certain lessons in algebra, it would be not less advantageous to carry on the study of these two sciences together, and to intermix them as much as possible. According as we advance in geometry, we find it necessary to combine together a greater number of relations, and algebra may be of great service in conducting us to our conclusions by the readiest and most easy method.

This work is divided into eight sections, four of which treat of plane geometry, and four of solid geometry.

The first section, entitled *first principles*, &c. contains the properties of straight lines which meet, those of perpendiculars, the theorem upon the sum of the angles of a triangle, the theory of parallel lines, &c.

The second section, entitled *the circle*, treats of the most simple properties of the circle, and those of chords, of tangents and of the measure of angles, by the arcs of a *circle*.

These two sections are followed by the resolution of certain problems relating to the construction of figures.

The third section, entitled the *proportions of figures*, contains the measure of surfaces, their comparison, the properties of a right-angled triangle, those of equiangular triangles, of similar

figures, &c. We shall be found fault with perhaps for having blended the properties of lines with those of surfaces; but in this we have followed pretty nearly the example of Euclid, and this order cannot fail of being good, if the propositions are well connected together. This section also is followed by a series of problems relating to the objects of which it treats.

The fourth section treats of *regular polygons and of the measure of the circle.* Two lemmas are employed as the basis of this measure, which is otherwise demonstrated after the manner of Archimedes. We have then given two methods of approximation for squaring the circle, one of which is that of James Gregory. This section is followed by an appendix, in which we have demonstrated that the circle is greater than any rectilineal figure of the same perimeter.

The first section of the second part contains the properties of *planes* and of *solid angles.* This part is very necessary for the understanding of solids and of figures in which different planes are considered. We have endeavoured to render it more clear and more rigorous than it is in common works.

The second section of the second part treats of *polyedrons* and of their measure. This section will be found to be very different from that relating to the same subject in other elements; we have thought we ought to present it in a manner entirely new.

The third section of this part is an abridged treatise *of the sphere and of spherical triangles.* This treatise does not ordinarily make a part of the elements of geometry; still we have thought it proper to consider so much of it as may form an introduction to spherical trigonometry.

The fourth section of the second part treats of *the three round bodies,* which are the sphere, the cone and the cylinder. The measure of the surfaces and solidities of these bodies is determined by a method analogous to that of Archimedes, and founded, as to surfaces, upon the same principles which we have endeavoured to demonstrate under the name of *preliminary lemmas.*

INTRODUCTION.

In order to abridge the language of geometry particular signs are substituted for the words which most frequently occur; and when we are employed upon any number or magnitude without considering its particular value, but merely with a view to indicate its relation to other magnitudes, or the operations to which it is to be subjected, we distinguish it by a letter of the alphabet, which thus becomes an abridged name for this magnitude.

I. $+$ signifies plus or added to.

The expression $A + B$ indicates the sum which results from the magnitude represented by the letter A being added to that represented by B, or A plus B.

$-$ signifies minus.

$A - B$ denotes what remains after the magnitude represented by B has been subtracted from that represented by A.

$\times$ signifies multiplied by.

$A \times B$ indicates the product arising from the magnitude represented by A being multiplied by the magnitude represented by B, or A multiplied by B. This product is also sometimes denoted by writing the letters one after the other without any sign, thus AB signifies the same as $A \times B$.

The expression $A \times (B + C - D)$ represents the product of A by the quantity $B + C - D$, the magnitudes included within the parenthesis being considered as one quantity.

$\frac{A}{B}$ indicates the quotient arising from the magnitude represented by A being divided by that represented by B, or A divided by B.

$A = B$ signifies that the magnitude represented by A is equal to that represented by B, or A equal to B.

$A > B$ signifies that the magnitude represented by A exceeds that represented by B, or A greater than B.

$A < B$ signifies A less than B.

$2A$, $3A$, &c., indicate double, triple, &c., of the magnitude represented by A.

II. When a number is multiplied by itself, the result is the *second power*, or *square*, of this number; 5×5, or 25, is the second power, or square, of 5.

The second power therefore is the product of two equal factors; each of these factors is the square root of the product; 5 is the square root of 25.

If the second power be multiplied by its root, the result is the *third power* or *cube*; 5×25 or 125 is the third power of 5.

The third power is a product formed by the multiplication of three equal factors; each of these factors is the *cube root* of this product; 125 is the product of 5 multiplied twice by itself, or $5 \times 5 \times 5$; and 5 is the cube root of 125.

In general A^2, being an abbreviation of $A \times A$, indicates the second power or square of A.

$\sqrt{A}$ indicates the square root of A, or the number which being multiplied by itself produces the number represented by A.

A^3, being an abbreviation of $A \times A \times A$, indicates the third power or cube of A.

$\sqrt[3]{A}$ indicates the cube root of A, or the number which, being multiplied twice by itself, produces the number A.

The square of a line AB is denoted by $\overline{AB}^2$.

The square root of a product $A \times B$ is represented by $\sqrt{A \times B}$.

All numbers are not perfect squares or perfect cubes, that is, they have not square roots or cube roots which can be exactly expressed; 19, for example, as it is between 16, the square of 4, and 25, the square of 5, has for its root a number comprehended between 4 and 5, but which cannot be exactly assigned.

In like manner 89, which is between 64, the cube of 4, and 125, the cube of 5, has for its cube root a number between 4 and 5, but which cannot be exactly assigned. Algebra furnishes methods for approximating, as nearly as we please, the roots of numbers which are not perfect powers.

III. 1. When two proportions have a common ratio, it is evident that the two other ratios may be put into a proportion, since they are each equal to that which is common. If, for example, we have

$$A : B :: C : D,$$
$$E : F :: C : D,$$

then we shall have $\quad A : B :: E : F.$

2. When two proportions have the same antecedents, the consequents may be put into a proportion; for, if we have

$$A : B :: C : D,$$
$$A : E :: C : F,$$

by changing the place of the means, these proportions will become

$$A : C :: B : D,$$
$$A : C :: E : F;$$

whence $\quad B : D :: E : F,$

or $\quad B : E :: D : F.$

IV. Other changes, besides the transposition of terms, may be made among proportionals without destroying the equality of the product of the extremes to that of the means.

1. If to the consequent of a ratio we add the antecedent, and compare this sum with the antecedent, this last will be contained once more than it was in the first consequent; the new ratio then will be equal to the primitive ratio increased by unity. If the same operation be performed upon the two ratios of a proportion, there will evidently result from it two new ratios equal to each other, and consequently a new proportion.

Let there be, for example, the proportion,

$$4 : 6 :: 12 : 18,$$

we shall have $\quad 6 + 4 : 4 :: 18 + 12 : 12,$

or $\quad 10 : 4 :: 30 : 12.$

2. If from the consequent of a ratio we subtract the antecedent, and compare the difference with the antecedent, this last will be contained once less than it was in the first consequent; the new ratio then will be equal to the primitive ratio diminished by unity. If the same operation be performed upon the two ratios of a proportion, there will result from it two new ratios equal to each other, and consequently a new proportion.

From the proportion

$$4 : 6 :: 12 : 18,$$

we thus deduce $$6 - 4 : 4 :: 18 - 12 : 12,$$
or $$2 : 4 :: 6 : 12.$$

There being a proportion among any magnitudes whatever designated by the letters

$$A : B :: C : D,$$

we have, by the above changes,

$$B + A : A :: D + C : C,$$
$$B - A : A :: D - C : C.$$

If we change the place of the means in these results, they will become

$$B + A : D + C :: A : C$$
$$B - A : D - C :: A : C;$$

but, by the same change, the proportion

$$A : B :: C : D$$

gives also $$A : C :: B : D,$$
and, since the ratios $A : C$, $B : D$ are equal, we obtain

$$B + A : D + C :: A : C \text{ or } :: B : D,$$
$$B - A : D - C :: A : C \text{ or } :: B : D,$$

a result which may be thus enunciated.

In any proportion whatever the sum of the two first terms is to the sum of the two last, and the difference of the two first terms is to the difference of the two last, as the first is to the third, or as the second is to the fourth.

Moreover the two ratios $A : C$, $B : D$, being common to the two proportions above obtained, it follows that the other ratios of the same proportions are equal, and that consequently

$$B + A : D + C :: B - A : D - C,$$

or, by changing the place of the means,

$$B + A : B - A :: D + C : D - C;$$

that is, *the sum of the two first terms of a proportion is to their difference, as the sum of the two last is to their difference.*

For example,

$$6 + 4 : 6 - 4 :: 18 + 12 : 18 - 12,$$

or $$10 : 2 :: 30 : 6.$$

When the proportion

$$A : B :: C : D,$$

is changed into $\quad A : C :: B : D,$

A and B are the antecedents, C and D the consequents; and the proportions

$$B + A : D + C :: A : C \text{ or } :: B : D,$$
$$B - A : D - C :: A : C \text{ or } :: B : D,$$

answer to the following enunciation;

The sum of the antecedents of a proportion is to the sum of the consequents, and the difference of the antecedents is to the difference of the consequents, as one antecedent is to its consequent;

Whence it follows that *the sum of the antecedents is to their difference as the sum of the consequents is to their difference.*

If we have a series of equal ratios

$$A : B :: C : D :: E : F,$$

by considering only the two first, which form the proportion

$$A : B :: C : D,$$

we obtain by what precedes

$$A + C : B + D :: A : B;$$

and, since the third ratio $E : F$ is equal to the first $A : B$, we have

$$A + C : B + D :: E : F.$$

If we take the sum of the antecedents and that of the consequents in this last proportion, the result will be

$$A + C + E : B + D + F :: E : F \text{ or } :: A : B.$$

By proceeding in the same manner with any number of equal ratios, it will be seen that, *the sum of any number whatever of antecedents is to the sum of their consequents, as one antecedent is to its consequent.*

V. Let there be any two proportions

$$A : B :: C : D,$$
$$E : F :: G : H,$$

if we multiply them in *order*, that is, term by term, the products will form a proportion, thus

$$A \times E : B \times F :: C \times G : D \times H,$$

This is evident, since the new ratios $\frac{B \times F}{A \times E}$, $\frac{D \times H}{C \times G}$, are respectively the products of the primitive ratios

$$\frac{B}{A} \text{ and } \frac{F}{E}, \frac{D}{C} \text{ and } \frac{H}{G},$$

which are equal.

If we multiply the proportion

$$A : B :: C : D$$

by $$A : B :: C : D$$

we shall have (II) $$A^2 : B^2 :: C^2 : D^2,$$

whence it follows, that *the squares of four proportional quantities form a new proportion.*

By multiplying the proportion

$$A^2 : B^2 :: C^2 : D^2$$

by $$A : B :: C : D,$$

we shall have $$A^3 : B^3 :: C^3 : D^3,$$

that is, *the cubes of four proportional quantities form a new proportion.*

Particular words are often used with reference to the changes that are made in the order or magnitude of proportional quantities; as *permutando* or *alternando*, *by permutation* or *alternately*, when, with respect to four proportionals, it is inferred that the first is to the third as the second is to the fourth; *invertendo*, *by inversion*, when it is inferred that the second is to the first as the fourth is to the third; *componendo*, *by composition*, when it is inferred that the first together with the second is to the second, as the third together with the fourth is to the fourth; *dividendo*, *by division*, when it is inferred that the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth; *convertendo*, *by conversion*, when it is inferred that the first is to its excess above the second, as the third is to its excess above the fourth.

VI. When a proportion is said to exist among certain magnitudes, these magnitudes are supposed to be represented, or to be capable of being represented by numbers; if, for example, in the proportion

$$A : B :: C : D,$$

A, B, C, D, denote certain lines, we can always suppose one of these lines, or a fifth, if we please, to answer as a common measure to the whole, and to be taken for unity; then A, B, C, D, will each represent a certain number of units, entire or fractional, commen-

surable or incommensurable, and the proportion among the lines A, B, C, D, becomes a proportion in numbers.

Hence the product of two lines A and D, which is called also their *rectangle*, is nothing else than the number of linear units contained in A multiplied by the number of linear units contained in B; and we can easily conceive this product to be equal to that which results from the multiplication of the lines B and C.

The magnitudes A and B in the proportion

$$A : B :: C : D,$$

may be of one kind, as lines, and the magnitudes C and D of another kind, as surfaces; still these magnitudes are always to be regarded as numbers; A and B will be expressed in linear units, C and D in superficial units, and the product $A \times D$ will be a number, as also the product $B \times C$.

Indeed, in all the operations, which are made upon proportional quantities, it is necessary to regard the terms of the proportion as so many numbers, each of its proper kind; then we shall have no difficulty in conceiving of these operations and of the consequences which result from them.

ELEMENTS OF GEOMETRY.

Definitions and preliminary remarks.

1. GEOMETRY is a science which has for its object the measure of extension.

Extension has three dimensions, length, breadth, and thickness.

2. A *line* is length without breadth.

The extremities of a line are called *points*. A point therefore has no extension.

3. A *straight* or *right line* is the shortest way from one point to another.

4. Every line, which is neither a straight line nor composed of straight lines, is a *curved line*.

Thus *AB* (*fig.* 1) is a straight line, *ACDB* is a *broken* line, or one composed of right lines, and *AEB* is a curved line. Fig. 1.

5. A *surface* is that which has length and breadth, without thickness.

6. A *plane* is a surface, in which any two points being taken, the straight line joining those points lies wholly in that surface.

7. Every surface, which is neither a plane nor composed of planes, is a *curved surface*.

8. A *solid* is that, which unites the three dimensions of extension.

9. When two straight lines, *AB*, *AC* (*fig.* 2), meet, the quantity, whether greater or less, by which they depart from each other as to their position, is called an *angle*; the point of meeting or *intersection A*, is the vertex of the angle; the lines *AB*, *AC*, are its *sides*. Fig. 2.

An angle is sometimes denoted simply by the letter at the vertex, as *A*; sometimes by three letters, as *BAC*, or *CAB*, the letter at the vertex always occupying the middle place.

Angles, like other quantities, are susceptible of addition, subtraction, multiplication, and division; thus, the angle *DCE*
20. (*fig.* 20) is the sum of the two angles *DCB*, *BCE*, and the angle *DCB* is the difference between the two angles *DCE*, *BCE*.

3. 10. When a straight line *AB* (*fig.* 3) meets another straight line *CD* in such a manner that the adjacent angles *BAC*, *BAD*, are equal, each of these angles is called a *right angle*, and the line *AB* is said to be *perpendicular* to *CD*.

4. 11. Every angle *BAC* (*fig.* 4), less than a right angle, is an *acute angle*; and every angle, *DEF*, greater than a right angle is an *obtuse angle*.

5 12. Two lines are said to be *parallel* (*fig.* 5), when, being situated in the same plane and produced ever so far both ways, they do not meet.

13. A *plane figure* is a plane terminated on all sides by lines. If the lines are straight, the space which they contain is called
5. a *rectilineal figure*, or *polygon* (*fig.* 6), and the lines taken together make the *perimeter* of the polygon.

14. The polygon of three sides is the most simple of these figures, and is called a *triangle*; that of four sides is called a *quadrilateral*; that of five sides, a *pentagon*; that of six, a *hexagon*, &c.

. 15. A triangle is denominated *equilateral* (*fig.* 7), when the
3. three sides are equal, *isosceles* (*fig.* 8), when two only of its sides
). are equal, and *scalene* (*fig.* 9), when no two of its sides are equal.

16. A *right-angled* triangle is that which has one right angle. The side opposite to the right angle is called the *hypothenuse*.
0. Thus *ABC* (*fig.* 10) is a triangle right-angled at *A*, and the side *BC* is the hypothenuse.

17. Among quadrilateral figures we distinguish;

1. The *square* (*fig.* 11), which has its sides equal and its angles right angles, (See art. 80);

2. The *rectangle* (*fig.* 12), which has its angles right angles without having its sides equal (See art. above referred to);

3. The *parallelogram* (*fig.* 13), which has its opposite sides parallel;

4. The *rhombus* or *lozenge* (*fig.* 14), which has its sides equal without having its angles right angles;

5. The *trapezoid* (*fig.* 15), which has two only of its sides parallel.

18. A *diagonal* is a line which joins the vertices of two angles not adjacent, as *AC* (*fig.* 42). Fig. 42

19. An *equilateral* polygon is one which has all its sides equal; an *equiangular* polygon is one which has all its angles equal.

20. Two polygons are *equilateral with respect to each other*, when they have their sides equal, each to each, and placed in the same order, that is, when by proceeding round in the same direction the first in the one is equal to the first in the other, the second in the one to the second in the other, and so on. In a similar sense are to be understood two polygons *equiangular with respect to each other*. The equal sides in the first case, and the equal angles in the second, are called *homologous* (A).

21. An *Axiom* is a proposition, the truth of which is self-evident.

A *Theorem* is a truth which becomes evident by a process of reasoning called a *demonstration*.

A *Problem* is a question proposed which requires a solution.

A *Lemma* is a subsidiary truth employed in the demonstration of a theorem, or in the solution of a problem.

The common name of *Proposition* is given indifferently to theorems, problems and lemmas.

A *Corollary* is a consequence which follows from one or several propositions.

A *Scholium* is a remark upon one or more propositions which have gone before, tending to show their connexion, their restriction, their extension, or the manner of their application.

A *Hypothesis* is a supposition made either in the enunciation of a proposition, or in the course of a demonstration.

Axioms.

22. Two quantities, each of which is equal to a third, are equal to one another.

23. The whole is greater than its part.

24. The whole is equal to the sum of all its parts.

25. Only one straight line can be drawn between two points.

26. Two magnitudes, whether they be lines, surfaces or solids, are equal, when, being applied the one to the other, they coincide with each other entirely, that is, when they exactly fill the same space.

PART FIRST.

OF PLANE FIGURES.

SECTION FIRST.

First principles, or the properties of perpendicular, oblique, and parallel lines.

THEOREM.

27. All *right angles are equal.*

Demonstration. Let the straight line *CD* be perpendicular to
16. *AB* (*fig.* 16), and *GH* to *EF*, the angles *ACD*, *EGH*, will be equal.

Take the four distances *CA*, *CB*, *GE*, *GF*, equal to each other, the distance *AB* will be equal to the distance *EF*, and the line *EF* may be applied to *AB*, so that the point *E* will fall upon *A*, and the point *F* upon *B*. These two lines, thus placed, will coincide with each other throughout; otherwise there would be two straight lines between *A* and *B*, which is impossible (25). The point *G* therefore, the middle of *EF*, will fall upon the point *C*, the middle of *AB*. The side *GE* being thus applied to *CA*. the side *GH* will fall upon *CD*; for, let us suppose, if it be possible, that it falls upon a line *CK*, different from *CD*; since, by hypothesis (10), the angle $EGH = HGF$,
it follows that $ACK = KCB$.
But $ACK > ACD$,
and $KCB < BCD$;
besides, by hypothesis,
$ACD = BCD$;
hence $ACK > KCB$,
and the line *GH* cannot fall upon a line *CK* different from *CD*; consequently it falls upon *CD*, and the angle *EGH* upon *ACD*; therefore all right angles are equal.

THEOREM.

17. 28. *A straight line* CD (fig. 17), *which meets another straight line* AB, *makes with it two adjacent angles* ACD, BCD, *which, taken together, are equal to two right angles.*

Demonstration. At the point C, let CE be perpendicular to AB. The angle ACD is the sum of the angles ACE, ECD; therefore $ACD + BCD$ is the sum of the three angles ACE, ECD, BCD. The first of these is a right angle, and the two others are together equal to a right angle; therefore the sum of the two angles ACD, BCD, is equal to two right angles.

29. *Corollary* I. If one of the angles ACD, BCD, is a right angle, the other is also a right angle.

30. *Corollary* II. If the line DE (*fig.* 18) is perpendicular to AB; reciprocally, AB is also perpendicular to DE. Fig. 18.

For, since DE is perpendicular to AB, it follows that the angle ACD is equal to its adjacent angle DCB, and that they are both right angles. But, since the angle ACD is a right angle, it follows that its adjacent angle ACE is also a right angle; therefore the angle $ACE = ACD$, and AB is perpendicular to DE.

31. *Corollary* III. All the successive angles, BAC, CAD, DAE, EAF (*fig.* 34), formed on the same side of the straight line BF, are together equal to two right angles; for their sum is equal to that of the two angles BAM, MAF; AM being perpendicular to BF. Fig. 34.

THEOREM.

32. *Two straight lines, which have two points common, coincide throughout, and form one and the same straight line.*

Demonstration. Let the two points, which are common to the two lines, be A and B (*fig.* 19). In the first place it is evident that they must coincide entirely between A and B; otherwise, two straight lines could be drawn from A to B, which is impossible (25). Now let us suppose, if it be possible, that the lines, when produced, separate from each other at a point C, the one becoming CD, and the other CE. At the point C, let CF be drawn, so as to make the angle ACF, a right angle; then, ACD being a straight line, the angle FCD is a right angle (29); and, because ACE is a straight line, the angle FCE is a right angle. But the part FCE cannot be equal to the whole FCD; whence straight lines, which have two points common A and B, cannot separate the one from the other, when produced; therefore they must form one and the same straight line. Fig. 19.

THEOREM.

Fig. 20. 33. *If two adjacent angles* ACD, DCB (fig. 20), *are together equal to two right angles, the two exterior sides* AC, CB, *are in the same straight line.*

Demonstration. For if *CB* is not the line *AC* produced, let *CE* be that line produced; then, *ACE* being a straight line, the angles *ACD, DCE*, are together equal to two right angles (28); but, by hypothesis, the angles *ACD, DCB*, are together equal to two right angles, therefore $ACD + DCB = ACD + DCE$. Take away the common angle *ACD*, and there will remain the part *DCB* equal to the whole *DCE*, which is impossible; therefore *CB* is the line *AC* produced.

THEOREM.

Fig. 21. 34. *Whenever two straight lines* AB, DE (fig. 21), *cut each other, the angles opposite*† *to each other at the vertex are equal.*

Demonstration. Since *DE* is a straight line, the sum of the angles *ACD, ACE*, is equal to two right angles; and, since *AB* is a straight line, the sum of the angles *ACE, BCE*, is equal to two right angles; therefore $ACD + ACE = ACE + BCE$; from each of these take away the common angle *ACE*, and there will remain the angle *ACD* equal to its opposite angle *BCE*.

It may be demonstrated, in like manner, that the angle *ACE* is equal to its opposite angle *BCD*.

35. *Scholium.* The four angles, formed about a point by two straight lines which cut each other, are together equal to four right angles; for the angles *ACE, BCE*, taken together, are equal to two right angles; also the other angles *ACD, BCD*, are together equal to two right angles.

Fig. 22. In general, if any number of right lines, as *CA, CB* (*fig.* 22), &c., meet in the same point *C*, the sum of all the successive angles, *ACB, BCD, DCE, ECF, FCA*, will be equal to four right angles. For, if at the point *C*, four right angles be formed by two lines perpendicular to each other, they will comprehend the same space as the successive angles *ACB, BCD*, &c.

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† These are often called *vertical* angles.

THEOREM.

36. *Two triangles are equal, when two sides and the included angle of the one are equal to two sides and the included angle of the other, each to each.*

Demonstration. In the two triangles ABC, DEF (*fig.* 23), let the angle A be equal to the angle D, the side AB equal to the side DE, and the side AC equal to the side DF; the two triangles ABC, DEF, will be equal. Fig. 23.

Indeed the triangles may be so placed, the one upon the other, that they shall coincide throughout. If, in the first place, we apply the side DE to its equal AB, the point D will fall upon A, and the point E upon B. But, since the angle D is equal to the angle A, when the side DE is placed upon AB, the side DE will take the direction AC; moreover DF is equal to AC; therefore the point F will fall upon C, and the third side EF will exactly coincide with the third side BC; therefore the triangle DEF is equal to the triangle ABC (26).

37. *Corollary.* When, in two triangles, these three things are equal, namely, the angle $A = D$, the side $AB = DE$, and the side $AC = DF$, we may thence infer, that the other three are also equal, namely, the angle $B = E$, the angle $C = F$, and the side $BC = EF$.

THEOREM.

38. *Two triangles are equal, when a side and the two adjacent angles of the one, are equal to a side and the two adjacent angles of the other, each to each.*

Demonstration. Let the side BC (*fig.* 23) be equal to the side EF, the angle B equal to the angle E, and the angle C equal to the angle F; the triangle ABC will be equal to the triangle DEF. Fig. 23.

For, in order to apply the one to the other, let EF be placed upon its equal BC, the point E will fall upon B and the point F upon C. Then, because the angle E is equal to the angle B, the side ED will take the direction BA, and therefore the point D will be somewhere in BA; also because the angle F is equal to C, the side FD will take the direction CA, and therefore the point D will be somewhere in CA; whence the point D, which must be at the same time in the lines BA and CA, can only be at their intersection A; therefore the two triangles ABC,

DEF, coincide, the one with the other, and are equal in all respects.

39. *Corollary.* When, in two triangles, these three things are equal, namely, $BC = EF$, $B = E$, and $C = F$, we may thence infer that the other three are also equal, namely, $AB = DE$, $AC = DF$, and $A = D$.

THEOREM.

40. *One side of a triangle is less than the sum of the other two.*

Fig. 23. *Demonstration.* The straight line *BC* (*fig.* 23), for example, is the shortest way from *B* to *C* (3); *BC* therefore is less than $BA + AC$.

THEOREM.

Fig. 24. 41. *If from a point* O (fig. 24), *within a triangle* ABC, *there be drawn straight lines* OB, OC, *to the extremities of* BC, *one of its sides, the sum of these lines will be less than that of* AB, AC, *the two other sides.*

Demonstration. Let *BO* be produced till it meet the side *AC* in *D*; the straight line *OC* is less than $OD + DC$; to each of these add *BO*, and $BO + OC < BO + OD + DC$; that is

$$BO + OC < BD + DC.$$

Again, $BD < BA + AD$; to each of these add *DC*, and we shall have $BD + DC < BA + AC$. But it has just been shown, that $BO + OC < BD + DC$, much more then is

$$BO + OC < BA + AC.$$

THEOREM.

Fig. 25. 42. *If two sides* AB, AC (fig. 25), *of a triangle* ABC, *are equal to two sides* DE, DF, *of another triangle* DEF, *each to each; if, at the same time, the angle* BAC, *contained by the former, is greater than the angle* EDF, *contained by the latter; the third side* BC *of the first triangle, will be greater than the third side* EF *of the second.*

Demonstration. Make the angle $CAG = D$, take $AG = DE$, and join *CG*, then the triangle *GAC* is equal to the triangle *EDF* (36), and therefore $CG = EF$. Now there may be three cases, according as the point *G* falls without the triangle *ABC*, on the side *BC*, or within the triangle.

Case 1. Because $GC < GI + IC$, and $AB < AI + IB$, therefore $GC + AB < GI + AI + IC + IB$, that is,

$$GC + AB < AG + BC.$$

From one of these take away AB, and from the other its equal AG, and there remains $GC < BC$; therefore $EF < BC$.

Case II. If the point G (*fig.* 26) fall upon the side BC, then it is evident that GC, or its equal EF, is less than BC. Fig. 26.

Case III. If the point G (*fig.* 27) fall within the triangle BAC, then $AG + GC < AB + BC$ (41), therefore, taking away the equal quantities, AG, AB, we shall have $GC < BC$, or $EF < BC$. Fig. 27.

THEOREM.

43. *Two triangles are equal, when the three sides of the one are equal to the three sides of the other, each to each.*

Demonstration. Let the side $AB = DE$ (*fig.* 23), $AC = DF$, $BC = EF$; then the angles will be equal, namely, $A = D$, $B = E$, and $C = F$. Fig 23.

For, if the angle A were greater than the angle D, as the sides AB, AC, are equal to the sides DE, DF, each to each, the side BC would be greater than EF (42); and if the angle A were less than the angle D, then the side BC would be less than EF; but BC is equal to EF, therefore the angle A can neither be greater nor less than the angle D, that is, it is equal to it. In the same manner it may be proved, that the angle $B = E$, and that the angle $C = F$.

44. *Scholium.* It may be remarked, that equal angles are opposite to equal sides; thus, the equal angles A and D are opposite to the equal sides BC and EF.

THEOREM.

45. *In an isosceles triangle the angles opposite to the equal sides are equal.*

Demonstration. Let the side $AB = AC$ (*fig.* 28), then will the angle C be equal to B. Fig. 28.

Draw the straight line AD from the *vertex* A to the point D the middle of the *base* BC; the two triangles ABD, ADC, will have the three sides of the one, equal to the three sides of the other, each to each, namely, AD common to both, $AB = AC$, by hypothesis, and $BD = DC$, by construction; therefore (43) the angle B is equal to the angle C.

46. *Corollary.* An equilateral triangle is also equiangular, that is, it has its angles equal.

47. *Scholium.* From the equality of the triangles ABD, ACD, it follows, that the angle $BAD = DAC$, and that the angle $BDA = ADC$; therefore these two last are right angles. *Hence a straight line drawn from the vertex of an isosceles triangle, to the middle of the base, is perpendicular to that base, and divides the vertical angle into two equal parts.*

In a triangle that is not isosceles, any one of its sides may be taken indifferently for a base; and then its vertex is that of the opposite angle. In an isosceles triangle, the base is that side which is not equal to one of the others.

THEOREM.

48. *Reciprocally, if two angles of a triangle are equal, the opposite sides are equal, and the triangle is isosceles.*

29. *Demonstration.* Let the angle $ABC = ACB$ (*fig.* 29), the side AC will be equal to the side AB.

For, if these sides are not equal, let AB be the greater. Take $BD = AC$, and join DC. The angle DBC is, by hypothesis, equal to ACB, and the two sides DB, BC, are equal to the two sides AC, CB, each to each; therefore the triangle DBC is equal to the triangle ACB (36); but a part cannot be equal to the whole; therefore the sides AB, AC, cannot be unequal; that is, they are equal, and the triangle is isosceles.

THEOREM.

49. *Of the two sides of a triangle, that is the greater, which is opposite to the greater angle; and conversely, of the two angles of a triangle, that is the greater, which is opposite to the greater side.*

30. *Demonstration.* 1. Let the angle $C > B$ (*fig.* 30), then will the side AB, opposite to the angle C, be greater than the side AC, opposite to the angle B.

Draw CD making the angle $BCD = B$. In the triangle BDC, BD is equal to DC (48); but $AD + DC > AC$, and

$$AD + DC = AD + DB = AB, \text{ therefore } AB > AC.$$

2. Let the side $AB > AC$, then will the angle C, opposite to the side AB, be greater than the angle B, opposite to the side AC. For, if C were less than B, then according to what has just been demonstrated we should have $AB < AC$, which is contrary to the hypothesis; and if C were equal to B, then it would

follow, that $AC = AB$ (48), which is also contrary to the hypothesis; whence the angle C can be neither less than B, nor equal to it; it is therefore greater.

THEOREM.

50. *From a given point* A (fig. 31), *without a straight line* DE, *only one perpendicular can be drawn to that line.* Fig. 31.

Demonstration. If it be possible, let there be two AB and AC; produce one of them AB, so that $BF = AB$, and join CF.

The triangle CBF is equal to the triangle ABC. For the angle CBF is a right angle (29), as well as CBA, and the side $BF = BA$; therefore the triangles are equal (36), and the angle $BCF = BCA$. But BCA is, by hypothesis, a right angle; therefore BCF is also a right angle. But, if the adjacent angles BCA, BCF, are together equal to two right angles, ACF must be a straight line (33); and hence it would follow that two straight lines ACF, ABF, might be drawn between the same two points A and F, which is impossible (25); it is then equally impossible to draw two perpendiculars from the same point to the same straight line.

51. *Scholium.* Through the same point C (*fig.* 17), in the line AB, it is also impossible to draw two perpendiculars to that line; for, if CD and CE were these two perpendiculars, the angle DCB would be a right angle as well as BCE; and a part would be equal to the whole. Fig. 17

THEOREM.

52. *If from a point* A (fig. 31), *without a straight line* DE. *a perpendicnlar* AB *be drawn to that line, and also different oblique lines* AE, AC, AD, &c., *to different points of the same line;* Fig. 31

1. *The perpendicular* AB *is less than any one of the oblique lines;*

2. *The two oblique lines* AC, AE, *which meet the line* DE *on opposite sides of the perpendicular, and at equal distances* BC, BE, *from it, are equal to one another;*

3. *Of any two oblique lines* AC, AD, *or* AE, AD, *that which is more remote from the perpendicular is the greater.*

Demonstration. Produce the perpendicular AB, so that $BF = BA$, and join FC, FD.

1. The triangle BCF is equal to the triangle BCA; for the right angle $CBF = CBA$, the side CB is common, and the side $BF = BA$; therefore the third side CF is equal to the third side AC (36). But $AF < AC + CF$ (40), and AB half of AF is less than AC half of $AC + CF$, that is, the perpendicular is less than any one of the oblique lines.

2. If $BE = BC$, then, as AB is common to the two triangles ABE, ABC, and the right angle $ABE = ABC$, the triangle ABE is equal to the triangle ABC, and $AE = AC$.

3. In the triangle DFA, the sum of the sides AD, DF, is greater than the sum of the sides AC, CF (41); therefore AD half of $AD + DF$ is greater than AC half of $AC + CF$, and the oblique line, which is more remote from the perpendicular, is greater than that which is nearer.

53. *Corollary* I. The perpendicular measures the distance of any point from a straight line.

54. *Corollary* II. From the same point, there cannot be drawn three equal straight lines terminating in a given straight line; for, if this could be done, there would be on the same side of the perpendicular two equal oblique lines, which is impossible.

THEOREM.

Fig. 32. 55. *If from the point* C (fig. 32), *the middle of the straight line* AB, *a perpendicular* EF *be drawn;* 1. *each point in the perpendicular* EF *is equally distant from the two extremities of the line* AB; 2. *any point without the perpendicular is at unequal distances from the same extremities* A *and* B.

Demonstration. 1. Since $AC = CB$, the two oblique lines AD, DB, are drawn to points which are at the same distance from the perpendicular. They are therefore equal (52). The same reasoning will apply to the two oblique lines AE, EB, also to AF, FB, &c. Whence each point in the perpendicular EF is equally distant from the extremities of the line AB.

2. Let I be a point out of the perpendicular; join IA, IB, one of these lines must cut the perpendicular in D; join DB, then $DB = DA$. But the line $IB < ID + DB$ and

$$ID + DB = ID + DA = IA;$$

therefore $IB < IA$; that is, any point without the perpendicular is at unequal distances from the extremities of AB.

THEOREM.

56. *Two right angled triangles are equal, when the hypothenuse and a side of the one are equal to the hypothenuse and a side of the other, each to each.*

Demonstration. Let the hypothenuse $AC = DF$ (*fig.* 33), and the side $AB = DE$; the right-angled triangle ABC will be equal to the right-angled triangle DEF. Fig. 33

The proposition will evidently be true, if the third side BC be equal to the third side EF. If it be possible, let these sides be unequal, and let BC be the greater. Take $BG = EF$, and join AG; then the triangle ABG is equal to the triangle DEF, for the right angle B is equal to the right angle E, the side $AB = DE$ and the side $BG = EF$; therefore these two triangles being equal (36), $AG = DF$; and, by hypothesis, $DF = AC$; whence $AG = AC$. But AG cannot be equal to AC (52); therefore it is impossible that BC should be unequal to EF, that is, it is equal to it, and the triangle ABC is equal to the triangle DEF.

THEOREM.

57. *If two straight lines* AC, BD (fig. 35), *are perpendicular to a third* AB, *these two lines are parallel, that is, they will not meet, however far they are produced* (12). Fig. 35

Demonstration. If they could meet in a point O on one side or the other of the line AB, there would be two perpendiculars drawn from the same point O to the same straight line AB, which is impossible (50).

LEMMA.

58. *The straight line* BD (fig. 35), *being perpendicular to* AB, *if another straight line* AE *make with* AB *an acute angle* BAE, *the straight lines* BD, AE, *being produced sufficiently far, will meet.* Fig. 35

Demonstration. If any point F, taken in the direction AE, let fall upon AB the perpendicular FG; the point G will not fall upon A, since the angle FAB is less than a right angle; still less can it fall upon H in BA produced, for then there would be two perpendiculars KA, KH, let fall from the same point K upon the same straight line AH. The point G then must fall, as the figure represents it, in the direction AB.

Let there be taken in the line AE another point L at a distance AL greater than AF, and let there be drawn to AB the perpendicular LM; it may be shown, as in the preceding case, that the point M can neither fall upon G nor upon any point in the direction GA, so that the distance AM will be necessarily greater than AG.

It may be observed, moreover, that if the figure is constructed with care, and AL be taken double of AF, we shall find that AM is exactly double of AG; also, if AL be taken triple of AF, we shall find that AM is triple of AG, and in general there will always be the same ratio between AM and AG, that there is between AL and AF. From this proportion it follows, not only that the right line AE, must meet BD, if the two lines are produced sufficiently far, but also that we may even assign upon AE the distance of the point of meeting of the two lines. This distance will be the fourth term of the proportion,

$$AG : AB :: AF : x.$$

59. *Scholium.* The foregoing explanation, founded upon a relation which is not deduced from reasoning merely, and for which recourse is had to measures taken upon a figure accurately constructed, has not the same degree of strictness, as the other demonstrations of elementary geometry. It is given here only as a simple method, by which one may satisfy himself of the truth of the proposition. We shall resume the subject with a view to a rigorous demonstration in the third of the notes subjoined to these elements.

THEOREM.

Fig. 36. 60. *If two straight lines* AC, BD (fig. 36), *make with a third* AB *two interior angles* CAB, ABD, *the sum of which is equal to two right angles, the two lines* AC, BD, *are parallel.*

Demonstration. From the point G, the middle of AB, draw the straight line EGF perpendicular to AC; this line will be perpendicular to BD. Indeed the sum $GAE + GBD$ is, by hypothesis, equal to two right angles, and the sum $GBF + GBD$ is also equal to two right angles (28); taking therefore from each GBD we shall have the angle $GAE = GBF$. Besides, the angles AGE, BGF, are equal, being vertical angles; therefore the triangles AGE, BGF, have a side and the two adjacent angles of the one res-

pectively equal to a side and the two adjacent angles of the other; they are therefore equal (38), and the angle

$$BFG = AEG;$$

but the angle AEG is, by construction, a right angle, consequently the straight lines AC, BD, are perpendicular to the same straight line EF, therefore they are parallel (57).

THEOREM.

61. *If two straight lines* AI, BD (fig. 36) *make with a third line* AB *two interior angles* BAI, ABD, *the sum of which is less than two right angles, the lines* AI, BD, *produced, will meet.* Fig. 36.

Demonstration. Draw AC making the angle $CAB = ABF$, that is, in such a manner as to make the two angles CAB, ABD, together equal to two right angles, and finish the construction, as in the preceding theorem. Since the angle AEK is a right angle, AE is a perpendicular, and consequently less than the oblique line AK; whence in the triangle AEK the angle AKE, opposite to the side AE, is less than the right angle AEK, opposite to the side AK (49). Therefore the angle IKF, equal to the angle AKE, is less than a right angle, and the lines KI, FD, being produced, must meet (58).

62. *Scholium.* If the lines AM and BD make with AB two angles BAM, ABD, the sum of which is greater than two right angles, then the two lines AM, BD, would not meet above AB, but they would meet below it. For the two angles BAM, BAN, would together be equal to two right angles, as also the two angles ABD, ABF, and the four angles taken together would be equal to four right angles. But the sum of the two angles BAM, ABD, is greater than two right angles, therefore the sum of the two remaining ones BAN, ABF, is less than two right angles, and the two right lines AN, BF, being produced, must meet.

63. *Corollary.* Through a given point A only one line can be drawn parallel to a given line BD. For there can be only one line AC, which makes the sum of the two angles BAC, ABD, equal to two right angles; this is the parallel required, every other line AI or AM would make the sum of the interior angles less or greater than two right angles, therefore it would meet the line BD.

THEOREM.

Fig. 37. 64. *If two parallel straight lines* AB, CD (fig. 37), *meet a third line* EF, *the sum of the interior angles upon the same side* AGH, GHC, *will be equal to two right angles.*

Demonstration. If this sum were greater or less than two right angles, the two straight lines *AB*, *CD*, would meet on one side or the other of *EF*, and would not be parallel (61).

65. *Corollary* I. If *GHC* be a right angle, *AGH* will also be a right angle; therefore every line, which is perpendicular to one of the parallels, is also perpendicular to the other.

66. *Corollary* II. Since the sum $AGH + GHC$ is equal to two right angles, and the sum $GHD + GHC$ is also equal to two right angles, if we take away the common part *GHC*, we shall have the angle $AGH = GHD$. Besides, $AGH = BGE$, and

$$GHD = CHF \text{ (34)};$$

therefore the four acute angles *AGH*, *BGE*, *GHD*, *CHF*, are equal to each other; the same may be proved with respect to the four obtuse angles *AGE*, *BGH*, *GHC*, *DHF*. It may be observed, moreover, that by adding one of the four acute angles to one of the four obtuse angles, the sum will always be equal to two right angles.

67. *Scholium.* The angles of which we have been speaking, compared, two and two, take different names. We have already called the angles *AGH*, *GHC*, *interior upon the same side;* the angles *BGH*, *GHD*, have the same name; the angles *AGH*, *GHD*, are called *alternate-internal*, or simply *alternate;* the same may be said of the angles *BGH*, *GHC*. Lastly, we denominate *internal-external* the angles *EGB*, *GHD*, and *EGA*, *GHC*, and *alternate-external* *EGB*, *CHF*, and *AGE*, *DHF*. This being premised, we may regard the following propositions as already demonstrated.

1. The two interior angles upon the same side, taken together, are equal to two right angles.

2. The alternate-internal angles are equal, as also the internal-external, and the alternate-external.

Reciprocally, if in this second case, two angles of the same name are equal, we may infer that the lines to which they are referred are parallel. Let there be, for example, the angle $AGH = GHD$; since $GHC + GHD$ is equal to two right angles,

we have also $AGH + GHC$ equal to two right angles, therefore the lines AG, CH, are parallel (60.)

THEOREM.

68. *Two lines* AB, CD (fig. 38), *which are parallel to a third* *EF, are parallel to one another.* Fig. 38.

Demonstration. Draw PQR perpendicular to EF. Then, since AB is parallel to EF, the line PR will be perpendicular to AB (65); also, since CD is parallel to EF, the line PR will be perpendicular to CD. Consequently AB and CD are perpendicular to the same straight line PQ, therefore they are parallel (57).

THEOREM.

69. *Two parallel lines are throughout at the same distance from each other.*

Demonstration. The two parallels AB, CD (*fig.* 39), being given, if through two points taken at pleasure we erect upon AB the two perpendiculars EG, FH, the straight lines EG, FH, will be at the same time perpendicular to CD (65); moreover these straight lines will be equal to each other. Fig. 39.

For, by drawing GF, the angles GFE, FGH, considered with reference to the parallels AB, CD, being alternate-internal angles (67), are equal; also since the straight lines EG, FH, are perpendicular to the same straight line AB and consequently parallel to each other, the angles EGF, GFH, considered with reference to the parallels EG, FH, being alternate-internal angles, are equal. The two triangles then EFG, FGH, have a side and the two adjacent angles of the one equal to a side and the two adjacent angles of the other, each to each; these two triangles are therefore equal (38); and the side EG, which measures the distance of the parallels AB, CD, at the point E, is equal to the side FH, which measures the distance of the same parallels at the point F.

THEOREM.

70. *If two angles* BAC, DEF (fig. 40), *have their sides parallel, each to each, and directed the same way, these two angles will be equal.* Fig. 40.

Demonstration. Produce DE, if it be necessary, till it meet AC in G; the angle DEF is equal to DGC, because EF is parallel to GC (67); the angle DGC is equal to BAC, because DG is parallel to AB; therefore the angle DEF is equal to BAC.

71. *Scholium.* There is a restriction in this proposition, namely, that the side EF be directed the same way as AC, and ED the same way as AB; the reason is this; if we produce FE toward H, the angle DEH would have its sides parallel to those of the angle BAC, but the two angles would not be equal. In this case the angle DEH and the angle BAC would together make two right angles.

THEOREM.

72. *In every triangle the sum of the three angles is equal to two right angles.*

Fig. 41. *Demonstration.* Let ABC (*fig.* 41) be any triangle; produce the side CA toward D, and draw to the point A the straight line AE parallel to BC.

Because AE, CB, are parallel, the angles ACB, DAE, considered with reference to the line CAD, are equal, being internal-external angles (67); in like manner ABC, BAE, considered with reference to the line AB, are equal, being alternate-internal angles; consequently the three angles of the triangle ABC make the same sum as the three angles CAB, BAE, EAD; therefore this sum is equal to two right angles (31).

73. *Corollary* I. Two angles of a triangle being given, or only their sum, the third will be known by subtracting the sum of these angles from two right angles.

74. *Corollary* II. If two angles of one triangle are equal to two angles of another triangle, each to each, the third of the one will be equal to the third of the other, and the two triangles will be equiangular.

75. *Corollary* III. In a triangle there can be only one right angle; for if there were two, the third angle must be nothing; still less then can a triangle have more than one obtuse angle.

76. *Corollary* IV. In every right-angled triangle the sum of the acute angles is equal to a right angle.

77. *Corollary* V. Every equilateral triangle, as it must be also equiangular (45), has each of its angles equal to a third of two

right angles, so that if a right angle be expressed by unity, the angle of an equilateral triangle will be expressed by $\frac{2}{3}$.

78. *Corollary* VI. In every triangle *ABC* (*fig.* 41) the exterior angle *BAD* is equal to the two opposite interior angles *B* and *C*; for, *AE* being parallel to *BC*, the part *BAE* is equal to the angle *B*, and the other part *DAE* is equal to the angle *C* (67). Fig. 41.

THEOREM.

79. *The sum of all the interior angles of a polygon is equal to as many times two right angles as there are units in the number of sides minus two.*

Demonstration. Let *ABCDE* &c. (*fig.* 42) be the proposed polygon; if from the vertex of the angle *A* we draw to the vertices of the opposite angles the diagonals *AC*, *AD*, *AE*, &c., it is evident, that the polygon will be divided into five triangles, if it have seven sides, and into six, if it have eight, and in general into as many triangles wanting two, as the polygon has sides; for these triangles may be considered as having for their common vertex the point *A*, and for their bases the different sides of the polygon, except the two which form the angle *BAC*. We see, at the same time, that the sum of the angles of all these triangles does not differ from the sum of the angles of the polygon; therefore this last sum is equal to as many times two right angles, as there are triangles, that is, as there are units in the number of sides of the polygon minus two. Fig. 42.

80. *Corollary* I. The sum of the angles of a quadrilateral is equal to two right angles multiplied by 4 — 2, which makes four right angles; therefore, if all the angles of a quadrilateral are equal, each of them will be a right angle, which justifies the definition of a square and rectangle (17).

81. *Corollary* II. The sum of the angles of a pentagon is equal to two right angles multiplied by 5 — 2, which makes 6 right angles; therefore, when a pentagon is equiangular, each angle is equal to a fifth of six right angles, or $\frac{6}{5}$ of one right angle.

82. *Corollary* III. The sum of the angles of a hexagon is equal to 2 × (6 — 2), or 8, right angles; therefore, in an equiangular hexagon, each angle is the sixth of eight right angles, or $\frac{4}{3}$ of one right angle. The process may be easily extended to other polygons.

83. *Scholium.* If we would apply this proposition to polygons, which have any *reentering*† angles, each of these angles is to be considered as greater than two right angles. But, in order to avoid confusion we shall confine ourselves in future to those polygons, which have only *saliant* angles, and which may be called *convex polygons.* Every convex polygon is such, that a straight line, however drawn, cannot meet the perimeter in more than two points.

THEOREM.

84. *The opposite sides of a parallelogram are equal, and the opposite angles also are equal.*

Fig. 44. *Demonstration.* Draw the diagonal BD (*fig.* 44); the two triangles ADB, DBC, have the side BD common; moreover, on account of the parallels AD, BC, the angle $ADB = DBC$ (67), and on account of the parallels AB. CD, the angle $ABD = BDC$; therefore the two triangles ADB, DBC, are equal (38); consequently the side AB opposite to ADB is equal to the side DC opposite to the equal angle DBC, and likewise the third side AD is equal to the third side BC; therefore the opposite sides of a parallelogram are equal.

Again, from the equality of the same triangles it follows, that the angle $A = C$, and also that the angle ADC, composed of the two angles ADB, BDC, is equal to the angle ABC, composed of the two angles DBC, ABD; therefore the opposite angles of a parallelogram are equal.

85. *Corollary.* Hence two parallels AB, CD, comprehended between two other parallels AD, BC, are equal.

THEOREM.

Fig. 44. 86. *If, in a quadrilateral* ABCD (fig. 44), *the opposite sides are equal, namely,* AB = CD, *and* AD = CB, *the equal sides will be parallel, and the figure will be a parallelogram.*

Demonstration. Draw the diagonal BD; the two triangles ABD, BDC, have the three sides of the one equal to the three

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† A reentering angle is one whose vertex is directed inward, as Fig. 43. CDE (*fig.* 43), while a saliant angle has its vertex directed outward as ABC.

sides of the other, each to each, they are therefore equal, and the angle ADB opposite the side AB is equal to the angle DBC opposite to the side CD; consequently the side AD is parallel to BC (67). For a similar reason AB is parallel to CD; therefore the quadrilateral $ABCD$ is a parallelogram.

THEOREM.

87. *If two opposite sides* AB, CD (fig. 44), *of a quadrilateral are equal and parallel, the two other sides will also be equal and parallel, and the figure* ABCD *will be a parallelogram.* Fig. 44.

Demonstration. Let the diagonal BD be drawn; since AB is parallel to CD, the alternate angles ABD, BDC, are equal (67). Besides, the side $AB = CD$, and the side DB is common, therefore the triangle ABD is equal to the triangle DBC (36), and the side $AD = BC$, the angle $ADB = DBC$, and consequently AD is parallel to BC; therefore the figure $ABCD$ is a parallelogram.

THEOREM.

88. *The two diagonals* AC, DB (fig. 45), *of a parallelogram mutually bisect each other.* Fig. 45.

Demonstration. If we compare the triangle ADO with the triangle COB, we find the side $AD = CB$, and the angle

$$ADO = CBO \text{ (67)};$$

also the angle $DAO = OCB$; therefore these two triangles are equal (38), and consequently AO, the side opposite to the angle ADO, is equal to OC, the side opposite to the angle OBC; DO likewise is equal to OB.

89. *Scholium.* In the case of the rhombus, the sides AB, BC, being equal, the triangles AOB, OBC, have the three sides of the one equal to the three sides of the other, each to each, and are consequently equal; whence it follows, that the angle

$$AOB = BOC,$$

and that thus the two diagonals of a rhombus cut each other mutually at right angles.

SECTION SECOND.

Of the circle and the measure of angles.

DEFINITIONS.

90. THE *circumference of a circle* is a curved line all the points of which are equally distant from a point within called the *centre.*

The circle is the space terminated by this curved line*.

Fig. 46. 91. Every straight line *CA, CE, CD* (*fig.* 46), &c., drawn from the centre to the circumference is called a *radius* or *semidiameter*, and every straight line, as *AB*, which passes through the centre and is terminated each way by the circumference, is called a *diameter.*

By the definition of a circle the radii are all equal, and all the diameters also are equal and double of the radius.

92. An *arc* of a circle is any portion of its circumference, as *FHG.*

The *chord* or *subtense* of an arc is the straight line *FG*, which joins its extremities**.

93. A *segment* of a circle is the portion comprehended between an arc and its chord.

94. A *sector* is the part of a circle comprehended between an arc *DE* and the two radii *CD, CE,* drawn to the extremities of this arc.

Fig. 47. 95. A *straight line* is said to be *inscribed in a circle*, when its extremities are in the circumference of the circle, as *AB* (*fig.*47).

An *inscribed angle* is one whose vertex is in the circumference, and which is formed by two chords, as *BAC.*

An *inscribed triangle* is a triangle whose three angles have their vertices in the circumference of the circle, as *BAC.*

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* In common discourse the circle is sometimes confounded with its circumference ; but it will always be easy to preserve the exactness of these expressions by recollecting that the circle is a surface which has length and breadth, while the circumference is only a line.

** The same chord, as *FG*, corresponds to two arcs, and consequently to two segments ; but, in speaking of these, the smaller is always to be understood, when the contrary is not expressed.

And in general an *inscribed figure* is one, all whose angles have their vertices in the circumference of the circle. In this case, the circle is said to be *circumscribed* about the figure.

96. A *secant* is a line, which meets the circumference in two points, as *AB* (*fig.* 48). Fig. 48.

97. A *tangent* is a line which has only one point in common with the circumference, as *CD*.

The common point *M* is called *the point of contact*.

Also two circumferences are *tangents* to each other (*fig.* 59, 60), Fig. 59, 60. when they have only one point common.

A polygon is said to be *circumscribed* about a circle, when all its sides are tangents to the circumference; and in this case the circle is said to be *inscribed* in the polygon.

THEOREM.

98. *Every diameter* AB (fig. 49) *bisects the circle, and its circumference.* Fig. 49.

Demonstration. If the figure *AEB* be applied to *AFB*, so so that the base *AB* may be common to both, the curved line *AEB* must fall exactly upon the curved line *AFB*; otherwise, there would be points in the one or the other unequally distant from the centre, which is contrary to the definition of a circle.

THEOREM.

99. *Every chord is less than the diameter.*

Demonstration. If the radii *CA*, *CD* (*fig.* 49), be drawn Fig. 49. from the centre to the extremities of the chord *AD*, we shall have the straight line $AD < AC + CD$, that is, $AD < AB$ (91).

100. *Corollary.* Hence the greatest straight line that can be inscribed in a circle is equal to its diameter.

THEOREM.

101. *A straight line cannot meet the circumference of a circle in more than two points.*

Demonstration. If it could meet it in three, these three points being equally distant from the centre, there might be three equal straight lines drawn from a given point to the same straight line, which is impossible (54).

THEOREM.

50. 102. *In the same circle, or in equal circles, equal arcs are subtended by equal chords, and conversely, equal chords subtend equal arcs.*

Demonstration. The radius AC (*fig.* 50) being equal to the radius EO, and the arc AMD equal to the arc ENG; the chord AD will be equal to the chord EG.

For, the diameter AB being equal to the diameter EF, the semicircle $AMDB$ may be applied exactly to the semicircle $ENGF$, and then the curved line $AMDB$ will coincide entirely with the curved line $ENGF$; but, the portion AMD being supposed equal to the portion ENG, the point D will fall upon G; therefore the chord AD is equal to the chord EG.

Conversely, AC being supposed equal to EO, if the chord $AD = EG$, the arc AMD will be equal to the arc ENG.

For, if the radii CD, OG, be drawn, the two triangles ACD, EOG, will have the three sides of the one equal to the three sides of the other, each to each, namely, $AC = EO$, $CD = OG$ and $AD = EG$; therefore these triangles are equal (43); hence the angle $ACD = EOG$. Now, if the semicircle ADB be placed upon EGF, because the angle $ACD = EOG$, it is evident, that the radius CD will fall upon the radius OG, and the point D upon G, therefore the arc AMD is equal to the arc ENG.

THEOREM.

103. *In the same circle, or in equal circles, the greater arc is subtended by the greater chord; and, conversely, if the arc be less than half the circumference, the greater arc subtends the greater chord.*

50. *Demonstration.* Let the arc AH (*fig.* 50) be greater than AD, and let the chords AD and AH, and the radii CD, CH, be drawn. The two sides AC, CH, of the triangle ACH, are equal to the two sides AC, CD, of the triangle ACD, and the angle ACH is greater than ACD; hence the third side AH is greater than the third side AD (42), therefore the greater arc is subtended by the greater chord.

Conversely, if the chord AH be greater than AD, it may be inferred from the same triangles that the angle ACH is greater than ACD, and that thus the arc AH is greater than AD.

104. ***Scholium.*** The arcs, of which we have been speaking, are supposed to be less than a semicircumference; if they were greater, the contrary would be true; in this case, as the arc increases the chord would diminish, and the reverse; thus, the arc *AKBD* being greater than *AKBH*, the chord *AD* of the first is less than the chord *AH* of the second.

THEOREM.

105. *The radius* CG (fig. 51), *perpendicular to a chord* AB, *bisects this chord and the arc subtended by it* AGB. Fig. 51.

Demonstration. Draw the radii *CA*, *CB*; these radii are, with respect to the perpendicular *CD*, two equal oblique lines, therefore they are equally distant from the perpendicular (52), and *AD* = *DB*.

Again, since *AD* = *BD*, and *CG* is a perpendicular erected upon the middle of *AB*, each point in *CG* is at equal distances from *A* and *B* (55). The point *G* is one of these points; therefore *AG* = *GB*. But, if the chord *AG* is equal to the chord *GB*, the arc *AG* will be equal to the arc *GB* (102); therefore the radius *CG*, perpendicular to the chord *AB*, bisects the arc subtended by this chord in the point *G*.

106. *Scholium.* The centre *C*, the middle *D* of the chord *AB*, and the middle *G* of the arc subtended by this chord, are three points situated in the same straight line perpendicular to the chord. Now, two points in a straight line are sufficient to determine its position; therefore a straight line which passes through any two of these points must necessarily pass through the third; and must be perpendicular to the chord.

It follows also, that a perpendicular erected upon the middle of a chord passes through the centre, and the middle, of the arc subtended by that chord.

For this perpendicular is the same as that let fall from the centre upon the same chord, since they both pass through the middle of the chord (51).

THEOREM.

107. *The circumference of a circle may be made to pass through any three points,* A, B, C (fig. 52), *which are not in the same* Fig. 52

straight line, but the circumference of only one circle may be made to pass through the same points.

Demonstration. Join AB, BC, and bisect these two straight lines by the perpendiculars DE, FG; these perpendiculars will meet in a point O.

For the lines DE, FG, will necessarily cut each other, if they are not parallel. Let us suppose that they are parallel; the line AB perpendicular to DE will be perpendicular to FG (65), and the angle K will be a right angle; but BK, which is BD produced, is different from BF, since the three points A, B, C, are not in the same straight line; there are then two perpendiculars BF, BK, let fall from the same point upon the same straight line, which is impossible (50); therefore the perpendiculars DE, FG, will always cut each other in some point O.

Now the point O, considered with reference to the perpendicular DE, is at equal distances from the two points A and B (55); also this same point O, considered with reference to the perpendicular FG, is at equal distances from the two points B and C; hence the three distances OA, OB, OC, are equal; therefore the circumference, discribed from the centre O with the radius OB, will pass through the three points A, B, C.

It is thus proved, that the circumference of a circle may be made to pass through any three given points, which are not in the same straight line; it remains to show, that there is only one circle, which can be so described.

If there were another circle, the circumference of which passed through the three given points A, B, C, its centre could not be without the line DE (55), since, in this case, it would be at unequal distances from A and B; neither can it be without the line FG, for a similar reason; it will then be in both of these lines at the same time. But two lines can cut each other in only one point (32); there is therefore only one circle, whose circumference can pass through three given points.

108. *Corollary.* Two circumferenees can meet each other only in two points; for, if they had three points common, they would have the same centre, and would make one and the same circumference.

THEOREM.

109. *Two equal chords are at the same distance from the centre, and of two unequal chords the less is at the greater aistance from the centre.*

Demonstration 1. Let the chord $AB = DE$ (*fig.* 53). Bisect these chords by the perpendiculars CF, CG, and draw the radii CA, CD. Fig. 53.

The right-angled triangles CAF, DCG, have the hypothenuses CA, CD, equal; moreover the side AF, the half of AB, is equal to the side DG, the half of DE; the triangles then are equal (58), and consequently the third side CF is equal to the third side CG; therefore the two equal chords AB, DE, are at the same distance from the centre.

2. Let the chord AH be greater than DE, the arc AKH will be greater than the arc DME (103). Upon the arc AKH take the part $ANB = DME$, draw the chord AB, and let fall the perpendicular CF upon this chord, and the perpendicular CI upon AH; CF is evidently greater than CO, and CO than CI (52); for a still stronger reason $CF > CI$. But $CF = CG$, since the chords AB, DE, are equal. Therefore $CG > CI$, and of two unequal chords the less is at the greater distance from the centre.

THEOREM.

110. *The perpendicular* BD (fig. 54), *at the extremity of the radius* AC, *is a tangent to the circumference.* Fig. 54.

Demonstration. Since every oblique line CE is greater than the perpendicular CA (52), the point E is without the circle, and the line BD has only the point A in common with the circumference; there BD is a tangent (97).

111. *Scholium.* We can draw through a given point A only one tangent AD to the circumference; for, if we could draw another, it would not be a perpendicular to the radius CA, and with respect to this new tangent the radius CA would be an oblique line, and the perpendicular let fall from the centre upon this tangent would be less than CA; therefore this supposed tangent would pass into the circle and become a secant.

THEOREM.

Fig. 55. 112. *Two parallels* AB, DE (fig. 55), *intercept upon the circumference equal arcs* MN, PQ.

Demonstration. The proposition admits of three cases.

1. If the two parallels are secants, draw the radius *CH* perpendicular to the chord *MP*, it will also be perpendicular to its parallel *NQ* (64), and the point *H* will be at the same time the middle of the arc *MHP* and of *NHQ* (105); whence the arc $MH = HP$, and the arc $NH = HQ$; also

$$MH - NH = HP - HQ, \text{ that is, } MN = PQ.$$

Fig. 56. 2. If of the two parallels *AB*, *DE* (*fig.* 56), one be a secant and the other a tangent; to the point of contact *H* draw the radius *CH*; this radius will be perpendicular to the tangent *DE* (110), and also to its parallel *MP*. But, since *CH* is perpendicular to the chord *MP*, the point *H* is the middle of the arc *MHP*; therefore the arcs *MH*, *HP*, comprehended between the parallels *AB*, *DE*, are equal.

3. If the two parallels *DE*, *IL*, are tangents, the one at *H* and the other at *K*; draw the parallel secant *AB*, and we shall have, according to what has just been demonstrated, $MH = HP$, and $MK = KP$; therefore the entire arc $HMK = HPK$, and it is moreover evident, that each of these arcs is a semicircumference.

THEOREM.

113. *If the circumferences of two circles cut each other in two points, the line which passes through their centres will be perpendicular to the chord, which joins the points of intersection, and will bisect it.*

Fig. 57, 58. *Demonstration.* The line *AB* (*fig.* 57, 58), which joins the points of intersection, is a chord common to the two circles; and, if a perpendicular be erected upon the middle of this chord, it must pass through each of the centres *C* and *D* (105). But through two given points only one straight line can be drawn; therefore the straight line, which passes through the centres, will be perpendicular to the middle of the common chord.

THEOREM.

114. *If the distance of two centres is less than the sum of the radii, and if at the same time the greater radius is less than the sum of the smaller and the distance of the centres, the two circles will cut each other.*

Demonstration. In order that the intersection may take place, the triangle ACD (*fig.* 57, 58) must be possible. It is necessary Fig. 57, 58.
then. not only that CD (*fig.* 57) should be less than $AC + AD$, Fig 57.
but also that the greater radius AD (*fig.* 58) should be less than Fig. 58.
$AC + CD$. Now, while the triangle CAD can be constructed, it is clear that the circumferences described from the centres C and D will cut each other in A and B.

THEOREM.

115. *If the distance* CD (fig. 59) *of the centres of two circles* Fig. 59.
is equal to the sum of their radii CA, CD, *these two circles will touch each other externally.*

Demonstration. It is evident that they will have the point A common, but they can have no other, for in order that there may be two points common, it is necessary that the distance of the centres should be less than the sum of the radii (114).

THEOREM.

116. *If the distance* CD *of the centres of two circles is equal to the difference of their radii* CA, AD (fig. 60), *these two circles* Fig. 60.
will touch each other internally.

Demonstration. In the first place it is evident, that they will have the point A common; and they can have no other, for in order that they may have two points common, it is necessary that the greater radius AD should be less than the sum of the radius AC and the distance of the centres CD (114), which is contrary to the supposition.

117. *Corollary.* Hence, if two circles touch each other, either internally or externally, the centres and the point of contact are in the same straight line.

118. *Scholium.* All the circles, which have their centres in the straight line CD and whose circumferences pass through the point A, touch each other, and have only the point A common. And if through the point A we draw AE perpendicular to CD, the straight line AE will be a tangent common to all these circles.

THEOREM.

119. *In the same circle, or in equal circles, equal angles* ACB,
61. DCE (fig. 61), *the vertices of which are at the centre, intercept upon the circumference equal arcs* AB, DE.

Reciprocally, if the arcs AB, DE, *are equal, the angles* ACB, DCE, *also will be equal.*

Demonstration. 1. If the angle *ACB* is equal to the angle *DCE*, these two angles may be placed the one upon the other, and as their sides are equal, it is evident, that the point *A* will fall upon *D*, and the point *B* upon *E*. But in this case the arc *AB* must also fall upon the arc *DE*; for if the two arcs were not coincident, there would be points in the one or the other at unequal distances from the centre, which is impossible; therefore the arc $AB = DE$.

2. If we suppose $AB = DE$, the angle *ACB* will be equal to *DCE*; for, if these angles are not equal, let *ACB* be the greater, and let *ACI* be taken equal to *DCE*; and we have, according to what has just been demonstrated, $AI = DE$. But, by hypothesis, the arc $AB = DE$; we should consequently have $AI = AB$, or the part equal to the whole, which is impossible; therefore the angle $ACB = DCE$.

THEOREM.

120. *In the same circle, or in equal circles, if two angles at the*
62. *centre* ACB, DCE (fig. 62), *are to each other, as two entire numbers, the intercepted arcs* AB, DE, *will be to each other, as the same numbers, and we shall have this proportion;*

angle ACB : *angle* DCE : : *arc* AB : *arc* DE.

Demonstration. Let us suppose, for example, that the angles *ACB*, *DCE*, are to each other, as 7 to 4; or, which amounts to the same, that the angle *M*, which will serve as a common measure, is contained seven times in the angle *ACB*, and four times in the angle *DCE*. The partial angles *ACm*, *mCn*, *nCp*, &c., *DCx*, *xCy*, &c., being equal to each other, the partial arcs *Am*, *mn*, *np*, &c., *Dx*, *xy*, &c., will also be equal to each other (119), and the entire arc *AB* will be to the entire arc *DE*, as 7 is to 4. Now it is evident, that the same reasoning might be used, whatever numbers were substituted in the place of 7 and 4; therefore, if the ratio of the angles *ACB*, *DCE*, can be expressed by

entire numbers, the arcs AB, DE, will be to each other, as the angles ACB, DCE.

121. *Scholium.* Reciprocally, if the arcs AB, DE, are to each other, as two entire numbers, the angles ACB, DCE, will be to each other, as the same numbers, and we shall have always $ACB : DCE :: AB : DE$; for the partial arcs Am, mn, &c., Dx, xy, &c., being equal, the partial angles ACm, mCn, &c., DCx, xCy, &c., are also equal.

THEOREM.

122. *Whatever may be the ratio of two angles* ACB, ACD, (fig. 63), *these two angles will always be to each other, as the arcs* AB, AD, *intercepted between their sides and described from their vertices, as centres, with equal radii.* Fig. 63.

Demonstration. Let us suppose the less angle placed in the greater; if the proposition enunciated be not true, the angle ACB will be to the angle ACD, as the arc AB is to an arc greater or less than AD. Let this arc be supposed to be greater, and let it be represented by AO; we shall have,

$$\text{angle } ACB : \text{angle } ACD :: \text{arc } AB : \text{arc } AO.$$

Let us now imagine the arc AB to be divided into equal parts, of which each shall be less than DO, there will be at least one point of division between D and O; let I be this point, and join CI; the arcs AB, AI, will be to each other, as two entire numbers, and we shall have, by the preceding theorem,

$$\text{angle } ACB : \text{angle } ACI :: \text{arc } AB : \text{arc } AI.$$

Comparing these two proportions together, and observing, that the antecedents are the same, we conclude that the consequents are proportional (III)†, namely,

$$\text{angle } ACD : \text{angle } ACI :: \text{arc } AO : \text{arc } AI.$$

But the arc AO is greater than the arc AI; it is necessary then, in order that this proportion may take place, that the angle ACD should be greater than the angle ACI; but it is less; it is therefore impossible, that the angle ACB should be to the angle ACD, as the arc AB is to an arc greater than AD.

By a process of reasoning altogether similar, it may be shown, that the fourth term of the proportion cannot be less than AD;

† The reference by Roman numerals is to the Introduction.

therefore it is exactly AD, and we have the proportion

angle ACB : angle ACD : : arc AB : arc AD.

123. *Corollary.* Since the angle at the centre of a circle the arc intercepted between its sides have such a connexion, when one increases or diminishes in any ratio whatever, other increases or diminishes in the same ratio, we are autl ized to establish one of these magnitudes as the measure of other; thus we shall, in future, take the arc AB as the meas of the angle ACB. The only thing to be observed in the c parison of angles with each other is, that the arcs, which used to measure them, must be described with equal radii. T is to be understood in the preceding propositions.

124. *Scholium.* It may seem more natural to measure a qu tity by another quantity of the same kind, and upon this pri ple it would be convenient to refer all angles to the right an and thus, the right angle being the unit of measure, the a angle would be expressed, by a number comprehended betw 0 and 1, and an obtuse angle by a number between 1 an But this manner of expressing angles would not be the n convenient in practice. It has been found much more simpl measure them by arcs of a circle on account of the facilit making arcs equal to given arcs and for many other reas Besides, if the measure of angles by the arcs of a circle b some degree indirect, it is not the less easy to obtain, by m of them, the direct and absolute measure; for, if we com the arc, which is used as the measure of an angle, with fourth part of the circumference, we have the ratio of the gi angle to a right angle, which is the absolute measure.

125. *Scholium* II. All that has been demonstrated in the tl preceding propositions, for the comparison of angles with a is equally applicable to the purpose of comparing sectors arcs; for sectors are equal, when their arcs are equal, an general they are proportional to the angles; hence *two sec* ACB, ACD, *taken in the same circle or in equal circles, ar each other, as the arcs* AB, AD, *the bases of these sectors.*

It will be perceived therefore, that the arcs of a circle, wl are used as a measure of angles, will also serve as the meas of different sectors of the same circle or of equal circles.

THEOREM.

126. *The inscribed angle* BAD (fig. 64, 65), *has for its measure the half of the arc* BD *comprehended between its sides.* Fig. 64, 65.

Demonstration. Let us suppose, in the first place, that the centre of the circle is situated in the angle BAD (*fig.* 64); we draw the diameter AE, and the radii CB, CD. The angle BCE, being the exterior angle of the triangle ABC, is equal to the sum of the two opposite interior angles CAB, ABC. But, the triangle BAC being isosceles, the angle $CAB = ABC$; hence the angle BCE is double of BAC. The angle BCE, having its vertex at the centre, has for its measure the arc BE; therefore the angle BAC has for its measure the half of BE. For a similar reason the angle CAD has for its measure the half of ED; therefore $BAC + CAD$, or BAD, has for its measure the half $BE + ED$, or the half of BD. Fig. 64.

Let us suppose, in the second place, that the centre C (*fig.* 65), is situated without the angle BAD; then, the diameter AE being drawn, the angle BAE will have for its measure the half of BE, and the angle DAE the half of DE; hence their difference BAD will have for its measure the half of BE minus the half of ED, or the half of BD. Fig. 65.

Therefore every inscribed angle has for its measure the half of the arc comprehended between its sides.

127. *Corollary* I. All the angles BAC, BDC (*fig.* 66), &c., inscribed in the same segment, are equal; for they have each for their measure the half of the same arc BOC. Fig. 66.

128. *Corollary* II. Every angle BAD (*fig.* 67), inscribed in a semicircle, is a right angle; for it has for its measure the half of the semicircumference BOD, or the fourth of the circumference. Fig. 67.

To demonstrate the same thing in another way, draw the radius AC; the triangle BAC is isosceles, and the angle

$$BAC = ABC;$$

the triangle CAD is also isosceles, and the angle $CAD = ADC$; hence $BAC + CAD$, or $BAD = ABD + ADB$. But, if the two angles B and D of the triangle ABD are together equal to the third BAD, the three angles of the triangle will be equal to twice the angle BAD; they are also equal to two right angles; therefore the angle BAD is a right angle.

Fig. 66. 129. *Corollary* III. Every angle BAC (*fig.* 66), inscribed in a segment greater than a semicircle, is an acute angle; for it has for its measure the half of the arc BOC less than a semicircumference.

And every angle BOC, inscribed in a segment less than a semicircle, is an obtuse angle; for it has for its measure the half of the arc BAC greater than a semicircumference.

Fig. 68. 130. *Corollary* IV. The opposite angles A and C (*fig.* 68) of an inscribed quadrilateral $ABCD$ are together equal to two right angles; for the angle BAD has for its measure the half of the arc BCD, and the angle BCD has for its measure the half of the arc BAD; hence the two angles BAD, BCD, taken together, have for their measure the half of the circumference; therefore their sum is equal to two right angles.

THEOREM.

Fig. 69. 131. *The angle* BAC (fig. 69), *formed by a tangent and a chord, has for its measure the half of the arc* AMDC, *comprehended between its sides.*

Demonstration. At the point of contact A draw the diameter AD; the angle BAD is a right angle (110), and has for its measure the half of the semicircumference AMD; the angle DAC has for its measure the half of DC; therefore $BAD + DAC$, or BAC, has for its measure the half of AMD plus the half of DC, or the half of the whole arc $AMDC$.

It may be demonstrated, in like manner, that CAE has for its measure the half of the AC, comprehended between its sides.

Problems relating to the two first sections.

PROBLEM.

Fig. 70. 132. *To divide a given straight line* AB (fig. 70) *into two equal parts.*

Solution. From the points A and B, as centres, and with a radius greater than the half of AB, describe two arcs cutting each other in D; the point D will be equally distant from the points A and B; find in like manner, either above or below the line AB a second point E equally distant from the points A and

B; through the two points *D* and *E* draw the line *DE*; this line will divide the line *AB* into two equal parts in the point *C*.

For, the two points *D* and *E* being each equally distant, from the extremities *A* and *B*, they must both be in the perpendicular which passes through the middle of *AB*. But through two given points only one straight line can be drawn; therefore the line *DE* will be this perpendicular, which divides the line *AB* into two equal parts in the point *C*.

PROBLEM.

133. *From a given point* A (fig. 71), *in the line* BC, *to erect a perpendicular to this line.* Fig. 71.

Solution. Take the points *B* and *C*, at equal distances from *A*; and from *B* and *C*, as centres, with a radius greater than *BA*, describe two arcs cutting each other in *D*; draw *AD*, which will be the perpendicular required.

For the point *D*, being equally distant from *B* and *C*, must be in a perpendicular to the middle of *BC* (55); therefore *AD* is this perpendicular.

134. *Scholium.* The same construction will serve to make a right angle *BAD* at a given point *A* in a given line *BC*.

PROBLEM.

135. *From a given point* A (fig. 72) *without the straight line* BD, *to let fall a perpendicular upon this line.* Fig. 72.

Solution. From *A*, as a centre, with a radius sufficiently great, describe an arc cutting the line *BD* in two points *B* and *D*; then find a point *E*, equally distant from the points *B* and *D* (132), and draw *AE*, which will be the perpendicular required.

For the two points *A* and *E* are each equally distant from the points *B* and *D*; therefore the line *AE* is perpendicular to the middle of *BD*.

PROBLEM.

136. *At a given point* A (fig. 73) *in the line* AB, *to make an angle equal to a given angle* K. Fig. 73.

Solution. From the vertex *K*, as a centre, with any radius, describe an arc *IL* meeting the sides of the angle, and from the point *A*, as a centre, with the same radius, describe an indefinite

arc BO; from B, as a centre, with a radius equal to the chord LI, describe an arc cutting the arc BO in D; draw AD, and the angle DAB will be equal to the given angle K.

For the arcs BD, LI, have equal radii and equal chords; they are therefore equal (102), and the angle $BAD = IKL$.

PROBLEM.

137. *To bisect a given arc or angle.*

Fig. 74. *Solution* 1. If it is proposed to bisect the arc AB (*fig.* 74); from the points A and B, as centres, with the same radius, describe two arcs intersecting each other in D; through the point D and the centre C draw CD, which will divide the arc AB into two equal parts in the point E.

For, since the points C and D are each equally distant from the extremities A and B of the chord AB, the line CD is perpendicular to the middle of this chord; therefore it bisects it (105).

2. If it is proposed to bisect the angle ACB; from the vertex C, as a centre, describe the arc AB, and complete the construction, as above described. It is evident that the line CD will bisect the angle ACB.

138. *Scholium.* By the same construction, we may bisect each of the halves AE, EB, and thus, by successive subdivisions, we may divide an angle or arc into four, eight, sixteen, &c., equal parts.

PROBLEM.

Fig. 75. 139. *Through a given point* A (fig. 75), *to draw a straight line parallel to a given straight line* BC.

Solution. From the point A, as a centre, with a radius sufficiently great, describe the indefinite arc EO; from the point E, as a centre, with the same radius, describe the arc AF; take

$$ED = AF,$$

and draw AD, which will be the parallel required.

For, AE being joined, the alternate angles AEF, EAD, are equal; therefore AD, EF, are parallel (67).

PROBLEM.

Fig. 76. 140. *Two angles* A and B (fig. 76) *of a triangle being given, to find the third.*

Solution. Draw the indefinite line *DEF*; at the point *E* make the angle $DEC = A$, and the angle $CEH = B$; the remaining angle *HEF* will be the third angle required; for these three angles are together equal to two right angles.

PROBLEM.

141. *Two sides of a triangle* B and C (fig. 77) *being given, and the angle* A *contained by them, to construct the triangle.* Fig. 77

Solution. Draw the indefinite line *DE*, and make at the point *D* the angle *EDF* equal to the given angle *A*; then take $DG = B$, $DH = C$, and draw *GH*; *DGH* will be the triangle required.

PROBLEM.

142. *One side and two angles of a triangle being given, to construct the triangle.*

Solution. The two given angles will be either both adjacent to the given side, or one adjacent and the other opposite. In this last case, find the third angle (140); we shall thus have the two adjacent angles. Then draw the straight line *DE* (*fig.* 78) Fig. 78 equal to the given side, at the point *D* make the angle *EDF* equal to one of the adjacent angles, and at the point *E* the angle *DEG* equal to the other; the two lines *DF*, *EG*, will cut each other in *H*, and *DEH* will be the triangle required.

PROBLEM.

143. *The three sides* A, B, C (fig. 79), *of a triangle being given, to construct the triangle.* Fig. 79

Solution. Draw *DE* equal to the side *A*; from the point *E*, as a centre, with a radius equal to the second side *B*, describe an arc; from the point *D*, as a centre, with a radius equal to the third side *C*, describe another arc cutting the former in *F*; draw *DF*, *EF*, and *DEF* will be the triangle required.

144. *Scholium.* If one of the sides be greater than the sum of the other two, the arcs will not cut each other; but the solution will always be possible, when each side is less than the sum of other two.

PROBLEM.

145. *Two sides* A *and* B *of a triangle being given with the angle* C *opposite to the side* B, *to construct the triangle.*

Solution. The problem admits of two cases. 1. If the angle
Fig. 80. C (*fig.* 80) is a right angle, or an obtuse angle, make the angle EDF equal to the angle C; take $DE = A$, from the point E, as a centre, and with a radius equal to the given side B, describe an arc cutting the line DF in F; draw EF, and DEF will be the triangle required.

It is necessary, in this case, that the side B should be greater than A. for the angle C being a right or an obtuse angle, it is the greatest of the angles of the triangle, and the side opposite must consequently be the greatest of the sides.

Fig. 81. 2. If the angle C (*fig.* 81) is acute, and B greater than A, the construction is the same, and DEF is the triangle required.

Fig. 82. But if, while C (*fig.* 82) is acute, the side B is less than A, then the arc described from the centre E with the radius $EF = B$, will cut the side DF in two points F and G situated on the same side of D; there are therefore two triangles DEF, DEG, which equally answer the conditions of the problem.

146. *Scholium.* The problem would be in every case impossible. if the side B were less than the perpendicular let fall from E upon the line DF.

PROBLEM.

Fig. 83. 147. *The adjacent sides* A *and* B (fig. 83) *of a parallelogram being given together with the included angle* C, *to construct the parallelogram.*

Solution. Draw the line $DE = A$; make the angle $FDE = C$, and take $DF = B$; describe two arcs, one from the point F, as a centre, with the radius $FG = DE$, and the other from the point E, as a centre, with the radius $EG = DF$; to the point G, where the two arcs cut each other, draw FG, EG; and $DEGF$ will be the parallelogram required.

For, by construction, the opposite sides are equal, therefore the figure is a parallelogram (86), and it is formed with the given adjacent sides and included angle.

148. *Corollary.* If the given angle be a right angle, the figure will be a rectangle; and, if the adjacent sides are also equal. the figure will be a square.

PROBLEM.

149. *To find the centre of a given circle, or of a given arc.*

Solution. Take at pleasure three points *A*, *B*, *C* (*fig.* 84), in the circumference of the circle or in the given arc; join *AB* and *BC*, and bisect them by the perpendiculars *DE*, *FG*; the point *O*, in which these perpendiculars meet, is the centre sought. Fig. 84.

150. *Scholium.* By the same construction a circle may be found, the circumference of which will pass through three given points *A*, *B*, *C*, or in which a given triangle *ABC* may be inscribed.

PROBLEM.

151. *Through a given point, to draw a tangent to a given circle.*

Solution. If the given point *A* (*fig.* 85) be in the circumference, draw the radius *CA*, and through *A* draw *AD* perpendicular to *CA*, then *AD* will be the tangent sought (110). If the point *A* (*fig.* 86) be without the circle, join the point *A* and the centre by the straight line *AC*; bisect *AC* in *O*, and from *O*, as a centre, with the radius *OC*, describe an arc cutting the given circle in the point *B*; draw *AB*, and *AB* will be the tangent required. Fig. 85. Fig. 86.

For, if we draw *CB*, the angle *CBA* inscribed in a semicircle is a right angle (128); therefore *AB*, being a perpendicular at the extremity of the radius *CB*, is a tangent.

152. *Scholium.* The point *A* being without the circle, it is evident that there are always two equal tangents *AB*, *AD*, which pass through the point *A*; they are equal (56), because the right-angled triangles *CBA*, *CDA*, have the hypothenuse *CA* common, and the side $CB = CD$; therefore $AD = AB$, and at the same time the angle $CAD = CAB$.

PROBLEM.

153. *To inscribe a circle in a given triangle* ABC (fig. 87). Fig. 87.

Bisect the angles *A* and *B* of the triangle by the straight lines *AO* and *BO*, which will meet each other in *O*; from the point *O* draw the perpendiculars *OD*, *OE*, *OF*, to the three sides of the triangle; these lines will be equal to each other. For, by construction, the angle $DAO = OAF$, and the right angle $ADO = AFO$;

consequently the third angle AOD is equal to the third AOF. Besides, the side AO is common to the two triangles AOD, AOF; therefore, a side and the adjacent angles of the one being respectively equal to a side and the adjacent angles of the other, the two triangles are equal; hence $DO = OF$. It may be shown, in like manner, that the two triangles BOD, BOE, are equal; consequently $OD = OE$; therefore the three perpendiculars OD, OE, OF, are equal to each other.

Now, if from the point O, as a centre, and with the radius OD, we describe a circle, it is evident that this circle will be inscribed in the triangle ABC; for the side AB, perpendicular to the radius at its extremity, is a tangent. The same may be said of the sides BC, AC.

154. *Scholium.* The three lines, which bisect the three angles of a triangle, meet in the same point.

PROBLEM.

Fig 88, 89. 155. *Upon a given straight line* AB (fig. 88, 89) *to describe a segment capable of containing a given angle* C, *that is a segment such, that each of the angles, which may be inscribed in it, shall be equal to a given angle* C.

Solution. Produce AB toward D, make at the point B the angle $DBE = C$, draw BO perpendicular to BE, and GO perpendicular to AB, G being the middle of AB; from the point of meeting O, as a centre, and with the radius OB, describe a circle; the segment required will be AMB.

For, since BF is perpendicular to the radius at its extremity, BF is a tangent, and the angle ABF has for its measure the half of the arc AKB (131); besides, the angle AMB, as an inscribed angle, has also for its measure the half of the arc AKB; consequently the angle $AMB = ABF = EBD = C$; therefore each of the angles inscribed in the segment AMB is equal to the given angle C.

156. *Scholium.* If the given angle were a right angle, the segment sought would be a semicircle described upon the diameter AB.

PROBLEM.

157. *To find the numerical ratio of two given straight lines* AB, CD (fig. 90), *provided, however, these two lines have a common measure.* Fig. 90.

Solution. Apply the smaller *CD* to the greater *AB*, as many times as it will admit of, for example, twice with a remainder *BE*.

Apply the remainder *BE* to the line *CD*, as many times as it will admit of, for example, once with a remainder *DF*.

Apply the second remainder *DF* to the first *BE*, as many times as will admit of, once, for example, with a remainder *BG*.

Apply the third remainder *BG* to the second *DF*, as many times as it will admit of.

Proceed thus, till a remainder arises, which is exactly contained a certain number of times in the preceding.

This last remainder will be the common measure of the two proposed lines; and, by regarding it as unity, the values of the preceding remainders are easily found, and, at length, those of the proposed lines, from which their ratio in numbers is deduced.

If, for example, we find that *GB* is contained exactly twice in *FD*, *GB* will be the common measure of the two proposed lines. Let $GB = 1$, we have $FD = 2$; but *EB* contains *FD* once plus *GB*; therefore $EB = 3$; *CD* contains *EB* once plus *FD*; therefore $CD = 5$; *AB* contains *CD* twice plus *EB*; therefore $AB = 13$; consequently the ratio of the two lines *AB*, *CD*, is as 13 to 5. If the line *CD* be considered as unity, the line *AB* would be $\frac{13}{5}$; and, if the line *AB* be considered as unity, the line *CD* would be $\frac{5}{13}$.

158. *Scholium.* The method, now explained, is the same as that given in arithmetic for finding the common divisor of two numbers (*Arith.* 61), and does not require another demonstration.

It is possible, that, however far we continue the operation, we may never arrive at a remainder, which shall be exactly contained a certain number of times in the preceding. In this case the two lines have no common measure, and they are said to be *incommensurable.* We shall see, hereafter, an example of this in the ratio of the diagonal to the side of a square. But, although the exact ratio cannot be found in numbers, by neglecting the last remainder we may find an approximate ratio to a greater

6

or less degree of exactness, according as the operation is more or less extended.

PROBLEM.

Fig. 91. 159. *Two angles* A *and* B (fig. 91) *being given, to find their common measure, if they have one, and from this their ratio in numbers.*

Solution. Describe, with equal radii, the arcs *CD, EF*, which may be regarded as the measure of these angles; in order then to compare the arc *CD, EF*, proceed as in the preceding problem; for an arc may be applied to an arc of the same radius, as a straight line is applied to a straight line. We shall thus obtain a common measure of the arcs *CD, EF*, if they have one, and their ratio in numbers. This ratio will be the same as that of the given angles (122); if *DO* is the common measure of the arcs, *DAO* will be the common measure of the angles.

160. *Scholium.* We may thus find the absolute value of an angle by comparing the arc, which serves as its measure, with the whole circumference. If, for example, the arc *CD* is to the circumference as 3 to 25, the angle *A* will be $\frac{3}{25}$ of four right angles, or $\frac{12}{25}$ of one right angle.

It may happen, as we have seen with respect to straight lines, that arcs also, which are compared, have not a common measure; we can then obtain, for the angles, only an approximate ratio in numbers, more or less exact, according to the degree to which the operation is extended.

SECTION THIRD.

Of the proportions of figures.

DEFINITIONS.

161. I SHALL call those figures *equivalent* whose surfaces are equal.

Two figures may be equivalent, however dissimilar; thus a circle may be equivalent to a square, a triangle to a rectangle, &c.

The denomination of *equal* figures will be restricted to those, which being applied, the one to the other, coincide entirely; thus two circles having the same radius are equal; and two triangles

having the three sides of the one equal to the three sides of the other, each to each, are also equal.

162. Two figures are *similar*, which have the angles of the one equal to the angles of the other, each to each, and the *homologous sides* proportional. By homologous sides are to be understood those, which have the same position in the two figures, or which are adjacent to equal angles. The angles, which are equal in the two figures, are called *homologous angles.*

Equal figures are always similar, but similar figures may be very unequal.

163. In two different circles, *similar arcs, similar sectors, similar segments,* are such as correspond to equal angles at the centre. Thus, the angle A (*fig.* 92) being equal to the angle O, the arc BC is similar to the arc DE, the sector ABC to the sector ODE, &c. Fig. 92.

164. The *altitude* of a parallelogram is the perpendicular which measures the distance between the opposite sides AB, CD (*fig.* 93), considered as bases. Fig. 93.

The *altitude of a triangle* is the perpendicular AD (*fig.* 94), let fall from the vertex of an angle A to the opposite side taken for a base. Fig. 94.

The *altitude of a trapezoid* is the perpendicular EF (*fig.* 95) drawn between its two parallel sides AB, CD. Fig. 95.

165. The *area* and the surface of a figure are terms nearly synonymous. Area, however, is more particularly used to denote the superficial extent of the figure considered as measured, or compared with other surfaces.

THEOREM.

166. *Parallelograms, which have equal bases and equal altitudes, are equivalent.*

Demonstration. Let AB (*fig.* 96) be the common base of the two parallelograms $ABCD$, $ABEF$; since they are supposed to have the same altitude, the sides DC, FE, opposite to the bases, will be situated in a line parallel to AB (69). Now, by the nature of a parallelogram, $AD = BC$ (84), and $AF = BE$; for the same reason, $DC = AB$, and $FE = AB$; therefore $DC = FE$. If DC be taken from DE, there will remain CE; and if FE, equal to DC, be taken also from DE, there will remain DF; consequently $CE = DF$. Fig. 96.

Hence the triangles *DAF*, *CBE*, have the three sides of the one equal to the sides of the other, each to each; they are therefore equal (43.)

But, if from the quadrilateral *ABED* the triangle *ADF* be taken, there will remain the parallelogram *ABEF*; and, if from the same quadrilateral *ABED* the triangle *CBE*, equal to the former, be taken, there will remain the parallelogram *ABCD*; therefore the two parallelograms *ABCD*, *ABEF*, which have the same base and the same altitude, are equivalent.

167. *Corollary.* Every parallelogram *ABCD* is equivalent to a rectangle of the same base and altitude.

THEOREM.

Fig. 98. 168. *Every triangle* ABC (fig. 98) *is half of a parallelogram* ABCD *the same base and altitude.*

Demonstration. The triangles *ABC*, *ACD*, are equal (87), therefore each is half of the parallelogram *ABCD*.

169. *Corollary* I. A triangle *ABC* is half of a rectangle *BCEF* of the same base *BC* and the same altitude *AO*; for the rectangle *BCEF* is equivalent to the parallogram *ABCD* (167).

170. *Corollary* II. All triangles, which have equal bases and equal altitudes, are equivalent.

THEOREM.

171. *Two rectangles, which have the same altitude, are to each other as their bases.*

Fig. 99. *Demonstration.* Let *ABCD*, *AEFD* (*fig.* 99), be two rectangles, which have a common altitude *AD*; they are to each other as their bases *AB*, *AE*.

Let us suppose, in the first place, that the bases *AB*, *AE*, are commensurable, and that they are to each other, as the numbers 7 and 4, for example; if we divide *AB* into 7 equal parts, *AE* will contain four of these parts; erect, at each point of division, a perpendicular to the base, we shall thus form seven partial rectangles which will be equal to each other, since they will have the same base and the same altitude (166). The rectangle *ABCD* will contain seven partial rectangles, while *AEFD* will contain four; therefore the rectangle *ABCD* is to the rectangle *AEFD*, as 7 is to 4, or as *AB* is to *AE*. The same rea-

soning may be applied to any other ratio beside that of 7 to 4; hence, whatever be the ratio, provided it is commensurable, we have

$$ABCD : AEFD :: AB : AE.$$

Let us suppose, in the second place, that the bases AB, AE (*fig.* 100), are incommensurable; we shall have notwithstanding Fig. 100.

$$ABCD : AEFD :: AB : AE.$$

For, if this proportion be not true, the three first terms remaining the same, the fourth will be greater or less than AE. Let us suppose that it is greater, and that we have

$$ABCD : AEFD : AB : AO.$$

Divide the line AB into equal parts smaller than EO, and there will be at least one point of division I between E and O; at this point erect the perpendicular IK; the bases AB, AI, will be commensurable, and we shall have, according to what has just been demonstrated,

$$ABCD : AIKD :: AB : AI.$$

But we have, by hypothesis,

$$ABCD : AEFD :: AB : AO.$$

In these two proportions the antecedents are equal, therefore the consequents are proportional (III); that is

$$AIKD : AEFD :: AI : AO.$$

Now AO is greater than AI; it is necessary then, in order that this proportion may take place, that the rectangle $AEFD$ should be greater than $AIKD$; but it is less; therefore the proportion is impossible, and $ABCD$ cannot be to $AEFD$, as AB is to a line greater than AE.

By a process entirely similar it may be shown, that the fourth term of the proportion cannot be smaller than AE; consequently it is equal to AE.

Whatever therefore be the ratio of the bases, two rectangles $ABCD$, $AEFD$, of the same altitude, are to each other as their bases AB, AE.

THEOREM.

172. *Any two rectangles* ABCD, AEGF (fig. 101), *are to each other, as the products of their bases by their altitudes, that is,* Fig. 101

$$ABCD : AEGF :: AB \times AD : AE \times AF.$$

Demonstration. Having disposed the two rectangles in such a manner, that the angles at A shall be opposite to each other, produce the sides GE, CD, till they meet in H; the two rectangles $ABCD$, $AEHD$, have the same altitude AD; they are consequently to each other as their bases AB, AE. Likewise the two rectangles $AEHD$, $AEGF$, have the same altitude AE; these are therefore to each other as their bases AD, AF. We have thus the two proportions

$$ABCD : AEHD :: AB : AE,$$
$$AEHD : AEGF :: AD : AF.$$

Multiplying these proportions in order and observing, that the connecting term $AEHD$ may be omitted, being a multiplier common to the antecedent and consequent, we have

$$ABCD : AEGF :: AB \times AD : AE \times AF.$$

173. *Scholium.* We may take for the measure of a rectangle the product of its base by its altitude, provided that, by this product, we understand that of two numbers which are the number of linear units contained in the base, and the number of linear units contained in the altitude.

This measure, however, is not absolute, but relative; it supposes that we estimate, in a similar manner, another rectangle by measuring its sides by the same linear unit; we obtain thus a second product, and the ratio of these two products is equal to that of the rectangles, conformably to the proposition, which has just been demonstrated.

If, for example, the base of a rectangle A be three units and its altitude ten, the rectangle would be represented by the number 3×10, or 30, a number which, thus disconnected, has no meaning; but, if we have a second rectangle B, whose base is twelve and altitude seven units, this rectangle will be represented by the number 7×12, or 84. Whence the two rectangles A and B are to each other, as 30 to 84. If therefore it is agreed to take the rectangle A, as the unit of measure for surfaces, the rectangle B will have for its absolute measure $\frac{84}{30}$, that is, it will be equal to $\frac{84}{30}$ superficial units.

The more common and simple method is to take the square as the unit of surface; and that square has been preferred, whose side is the unit of length; the measure therefore, which we have regarded as simply relative, becomes absolute. The number 30, for example, by which we have measured the rec-

tangle *A*, represents 30 superficial units, or 30 of those squares, the side of each of which is equal to unity. This is illustrated by figure 102.

In geometry, the product of two lines often signifies the same thing as their *rectangle*, and this expression is introduced into arithmetic to denote the product of two unequal numbers, as that of *square* is used to express the product of a number by itself.

The squares of the numbers 1, 2, 3, &c., are 1, 4, 9, &c. Thus a double line gives a quadruple square, a triple line a square nine times as great, and so on.

THEOREM.

174. *The area of any parallelogram is equal to the product of its base by its altitude.*

Demonstration. The parallelogram *ABCD* (*fig.* 97) is equivalent to the rectangle *ABEF*, which has the same base *AB* and the same altitude *BE* (167); but this last has for its measure $AB \times BE$ (173); therefore $AB \times BE$ is equal to the area of the parallelogram *ABCD*. Fig. 97.

175. *Corollary.* Parallelograms of the same base are to each other as their altitudes, and parallelograms of the same altitude are to each other as their bases; for, *A*, *B*, *C*, being any three magnitudes whatever, we have generally $A \times C : B \times C :: A : B$.

THEOREM.

176. *The area of a triangle is equal to the product of its base by half its altitude.*

Demonstration. The triangle *ABC* (*fig.* 104) is half of the parallelogram *ABCE*, which has the same base *BC* and the same altitude *AD* (168); now the area of the parallelogram $= BC \times AD$ (174); therefore the area of the triangle $= \frac{1}{2}\, BC \times AD$, or $BC \times \frac{1}{2}\, AD$. Fig. 10

177. *Corollary.* Two triangles of the same altitude are to each other as their bases, and two triangles of the same base are to each other as their altitudes.

THEOREM.

178. *The area of a trapezoid* ABCD (fig. 105) *is equal to the product of its altitude* EF *by half the sum of its parallel sides* AB, CD. Fig. 10

Demonstration. Through the point I, the middle of the side CB, draw KL parallel to the opposite side AD, and produce DC till it meet KL in K.

In the triangles IBL, ICK, the side $IB = IC$, by construction; the angle $LIB = CIK$, and the angle $IBL = ICK$, since CK and BL are parallel (67); therefore these triangles are equal (38), and the trapezoid $ABCD$ is equivalent to the parallelogram $ADKL$, and has for its measure $EF \times AL$.

But $AL = DK$; and, since the triangle IBL is equal to the triangle KCI, the side $BL = CK$; therefore

$$AB + CD = AL + DK = 2AL;$$

thus AL is half the sum of the sides AB, CD; and consequently the area of the trapezoid $ABCD$ is equal to the product of the altitude EF by half the sum of the sides AB, CD, which may be expressed in this manner; $ABCD = EF \times \left(\frac{AB + CD}{2}\right)$.

179. *Scholium.* If through the point I, the middle of BC, IH be drawn parallel to the base AB, the point H will also be the middle of AD; for the figure $AHIL$ is a parallelogram, as well as $DHIK$, since the opposite sides are parallel; we have therefore $AH = IL$, and $DH = IK$; but $IL = IK$, because the triangles BIL, CIK, are equal; therefore $AH = DH$.

It may be remarked, that the line $HI = AL = \frac{AB + CD}{2}$; therefore the area of the trapezoid may be expressed also by $EF \times HI$; that is, it is equal to the product of the altitude of the trapezoid by the line joining the middle points of the sides which are not parallel.

THEOREM.

Fig. 106. 180. *If a line* AC (fig. 106) *is divided into two parts* AB, BC, *the square described upon the whole line* AC *will contain the square described upon the part* AB, *plus the square described upon the other part* BC, *plus twice the rectangle contained by the two parts* AB, BC; *which may be thus expressed,*

$$\overline{AC}^2 \text{ or } (AB + BC)^2 = \overline{AB}^2 + \overline{BC}^2 + 2AB \times BC.$$

Demonstration. Construct the square $ACDE$, take $AF = AB$, draw FG parallel to AC, and BH parallel to AE.

The square *ACDE* is divided into four parts; the first *ABIF* is the square described upon *AB*, since *AF* was taken equal to *AB*; the second *IGDH* is the square described upon *BC*; for, since $AC = AE$, and $AB = AF$, the difference $AC - AB = AE - AF$, which gives $BC = EF$; but, on account of the parallels, $IG = BC$, and $DG = EF$, therefore *HIGD* is equal to the square described upon *BC*. These two parts being taken from the whole square, there remain the two rectangles *BCGI*, *EFIH*, which have each for their measure $AB \times BC$; therefore the square described upon *AC*, &c.

181. *Scholium.* This proposition corresponds to that given in algebra for the formation of the square of a binomial, which is thus expressed,

$$(a + b)^2 = a^2 + 2ab + b^2.$$

THEOREM.

182. *If the line* AC (*fig.* 107) *is the difference of two lines* AB, BC, *the square described upon* AC *will contain the square of* AB, *plus the square of* BC, *minus twice the rectangle contained by* AB *and* BC; *that is,* $\overline{AC}^2$ *or* $(AB - BC)^2 = \overline{AB}^2 + \overline{BC}^2 - 2AB \times BC$. Fig. 107.

Demonstration. Construct the square *ABIF*, take $AE = AC$, draw *CG* parallel to *BI*, *HK* parallel to *AB*, and finish the square *EFLK*.

The two rectangles *CBIG*, *GLKD*, have each for their measure $AB \times BC$; if we subtract them from the whole figure *ABILKEA*, which has for its value $\overline{AB}^2 + \overline{BC}^2$, it is evident, that there will remain the square *ACDE*; therefore, if the line *AC*, &c.

183. *Scholium.* This proposition answers to the algebraic formula $(a - b)^2 = a^2 + b^2 - 2ab$.

THEOREM.

184. *The rectangle contained by the sum and difference of two lines is equal to the difference of their squares; that is*

$$(AB + BC) \times (AB - BC) = \overline{AB}^2 - \overline{BC}^2 \text{ (fig. 108).}$$ Fig. 108.

Demonstration. Construct upon *AB* and *AC* the squares *ABIF*, *ACDE*; produce *AB* making $BK = BC$, and complete the rectangle *AKLE*.

The base AK of the rectangle is the sum of the two lines AB, BC, its altitude AE is the difference of these lines; therefore the rectangle $AKLE = (AB + BC) \times (AB - BC)$. But this same rectangle is composed of two parts $ABHE + BHLK$, and the part $BHLK$ is equal to the rectangle $EDGF$, for $BH = DE$, and $BK = EF$; consequently $AKLE = ABHE + EDGF$. Now these two parts form the square $ABIF$, minus the square $DHIG$ which is the square described upon BC; therefore

$$(AB + BC) \times (AB - BC) = \overline{AB}^2 - \overline{BC}^2.$$

185. *Scholium.* This proposition agrees with the algebraic formula $(a + b) \times (a - b) = a^2 - b^2$ (Alg. 34).

THEOREM.

186. *The square described upon the hypothenuse of a right-angled triangle is equal to the sum of the squares described upon the two other sides.*

Fig. 109. *Demonstration.* Let ABC (*fig.* 109) be a triangle right-angled at A. Having constructed squares upon the three sides, let fall, from the right angle upon the hypothenuse, the perpendicular AD, which produce to E, and draw the diagonals AF, CH.

The angle ABF is composed of the angle ABC plus the right angle CBF; and the angle HBC is composed of the same angle ABC plus the right angle ABH; hence the angle $ABF = HBC$. But $AB = BH$, being sides of the same square; and $BF = BC$, for the same reason; consequently the triangles ABF, HBC, have two sides and the included angle of the one respectively equal to two sides and the included angle of the other; they are therefore equal (36).

The triangle ABF is half of the rectangle BE†, which has the same base BF and the same altitude BD (169). Also the triangle HBC is half of the square AH; for, the angle BAC being a right angle as well as BAL, AC and AL are in the same straight line parallel to HB: hence the triangle HBC and the square AH have the same base BH, and the same altitude AB; therefore the triangle is half of the square.

† An abridged expression for $BDEF$.

It has already been proved, that the triangle ABF is equal to the triangle HBC; consequently the rectangle $BDEF$, double of the triangle ABF, is equivalent to the square AH, double of the triangle HBC. It may be demonstrated, in the same manner, that the rectangle $CDEG$ is equivalent to the square AI; but the two rectangles $BDEF$, $CDEG$, taken together, make the square $BCGF$; therefore the square $BCGF$, described upon the hypothenuse, is equal to the sum of the squares $ABHL$, $ACIK$, described upon the two other sides; or, $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$.

187. *Corollary* I. The square of one of the sides of a right-angled triangle is equal to the square of the hypothenuse minus the square of the other side; or $\overline{AB}^2 = \overline{BC}^2 - \overline{AC}^2$.

188. *Corollary* II. Let $ABCD$ (*fig.* 118) be a square, AC its diagonal; the triangle ABC being right-angled and isosceles, we have $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = 2\overline{AB}^2$; therefore *the square described upon the diagonal* AC *is double of the square described upon the side* AB. Fig. 11

This property may be rendered sensible by drawing, through the points A and C, parallels to BD, and through the points B and D parallels to AC; a new square $EFGH$ is thus formed which is the square of AC. It is manifest that $EFGH$ contains eight triangles, each of which is equal ABE, and that $ABCD$ contains four of them; therefore the square $EFGH$ is double of $ABCD$.

Since $\overline{AC}^2 : \overline{AB}^2 :: 2 : 1$, we have, by extracting the square root, $AC : AB :: \sqrt{2} : 1$; therefore *the diagonal of a square is incommensurable with its side* (Alg. 99).

This will be more fully developed hereafter.

189. *Corollary* III. It has been demonstrated, that the square AH (*fig.* 109) is equivalent to the rectangle $BDEF$; now, on account of the common altitude BF, the square $BCGF$ is to the rectangle $BDEF$ as the base BC is to the base BD; therefore Fig. 10

$$\overline{BC}^2 : \overline{AB}^2 :: BC : BD,$$

or, *the square of the hypothenuse is to the square of one of the sides of the right angle as the hypothenuse is to the segment adjacent to this side.* We give the name of *segment* to that part of the hypothenuse cut off by the perpendicular let fall from the right angle; thus BD is the segment adjacent to the side AB, and DC the segment adjacent to the side AC. We have likewise

$$\overline{BC}^2 : \overline{AC}^2 : : BC : CD.$$

190. *Corollary* IV. The rectangles *BDEF*, *DCGE*, having also the same altitude *DE*, are to each other as their bases *BD*, *CD*. Now these rectangles are equivalent to the squares *AH*, *AI*, therefore,

$$\overline{AB}^2 : \overline{AC}^2 : : BD : DC,$$

or, *the squares of the two sides of a right angle are to each other, as the segments of the hypothenuse adjacent to these sides.*

THEOREM.

ig. 110. 191. *In a triangle* ABC (fig. 110), *if the angle* C *be acute, the square of the side opposite to it will be less than the sum of the squares of the sides containing it, and,* AD *being drawn perpendicular to* BC, *the difference will be equal to double the rectangle* BC × CD, *or,*

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 - 2BC \times CD.$$

Demonstration. The proposition admits of two cases. 1. If the perpendicular fall within the triangle *ABC*, we shall have $BD = BC - CD$, and consequently (182)

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 - 2BC \times CD;$$

adding $\overline{AD}^2$ to each member, we have

$$\overline{AD}^2 + \overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 + \overline{AD}^2 - 2BC \times CD;$$

but the right-angled triangles *ABD*, *ADC*, give $\overline{AD}^2 + \overline{BD}^2 = \overline{AB}^2$, $\overline{CD}^2 + \overline{AD}^2 = \overline{AC}^2$, therefore

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2BC \times CD.$$

2. If the perpendicular *AD* fall without the triangle *ABC*, we shall have $BD = CD - BC$, and consequently (182)

$$\overline{BD}^2 = \overline{CD}^2 + \overline{BC}^2 - 2BC \times CD;$$

adding to each $\overline{AD}^2$, and we shall obtain, as before,

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2BC \times CD.$$

THEOREM.

ig. 111. 192. *In a triangle* ABC (fig. 111), *if the angle* C *be obtuse, the square of the side opposite to it will be greater than the sum of the

squares of the sides containing it, and, AD *being drawn perpendicular to* BC *produced, the difference will be equal to double the rectangle* BC × CD, *or,*

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 + 2BC \times CD.$$

Demonstration. The perpendicular cannot fall within the triangle; for if it should fall, for example, upon *E*, the triangle *ACE* would have at the same time a right angle *E* and an obtuse angle *C*, which is impossible (75); consequently it falls without, and we have $BD = BC + CD$, and from this (180)

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 + 2BC \times CD.$$

Adding to each term $\overline{AD}^2$, and making the reductions as in the preceding theorem, we obtain

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2BC \times CD.$$

193. *Scholium.* The right-angled triangle is the only one in which the sum of the squares of two of the sides is equal to the square of the third; for, if the angle contained by their sides be acute, the sum of their squares will be greater than the square of the side opposite; if it be obtuse, the reverse will be true.

THEOREM.

194. *In any triangle* ABC (fig. 112), *if we draw from the vertex to the middle of the base the line* AE, *we shall have* Fig. 112

$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{AE}^2 + 2\overline{EB}^2.$$

Demonstration. Let fall the perpendicular *AD* upon the base *BC*, the triangle *AEC* will give (191),

$$\overline{AC}^2 = \overline{AE}^2 + \overline{EC}^2 - 2EC \times ED;$$

the triangle *ABE* will give (192),

$$\overline{AB}^2 = \overline{AE}^2 + \overline{EB}^2 + 2EB \times ED;$$

therefore, by adding the corresponding members, and observing that $EB = EC$, we shall have

$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{AE}^2 + 2\overline{EB}^2.$$

195. *Corollary. In every parallelogram the sum of the squares of the sides is equal to the sum of the squares of the diagonals.*

For the diagonals *AC*, *BD* (*fig.* 113), mutually bisect each other in the point *E* (88), and the triangle *ABC* gives Fig. 113

$$\overline{AB}^2 + \overline{BC}^2 = 2\overline{AE}^2 + 2\overline{BE}^2;$$

the triangle ADC gives likewise

$$\overline{AD}^2 + \overline{DC}^2 = 2\overline{AE}^2 + 2\overline{DE}^2;$$

adding the corresponding members and observing that $BE = DE$, we have

$$\overline{AB}^2 + \overline{AD}^2 + \overline{DC}^2 + \overline{BC}^2 = 4\overline{AE}^2 + 4\overline{DE}^2.$$

But $4\overline{AE}^2$ is the square of $2AE$ or of AC; and $4\overline{DE}^2$ is the square BD; therefore the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

THEOREM.

Fig 114. 196. *The line* DE (fig. 114), *drawn parallel to the base of a triangle* ABC, *divides the sides* AB, AC, *proportionally; so that* AD : DB : : AE : EC.

Demonstration. Join BE and DC; the two triangles BDE, DEC, have the same base DE; they have also the same altitude, since the vertices B and C are situated in a parallel to the base; therefore the triangles are equivalent (170).

The triangles ADE, BDE, of which the common vertex is E, have the same altitude, and are to each other as their bases AD, DB (177); thus,

$$ADE : BDE :: AD : DB.$$

The triangles ADE, DEC, of which the common vertex is D, have also the same altitude, and are to each other as their bases AE, EC; that is, $ADE : DEC :: AE : EC$.

But it has been shown that the triangle $BDE = DEC$; therefore, on account of the common ratio in the two proportions (III),

$$AD : DB :: AE : EC.$$

197. *Corollary* I. We obtain from the above theorem by *composition* (IV)

$$AD + DB : AD :: AE + EC : AE,$$

or

$$AB : AD :: AC : AE,$$

also

$$AB : BD :: AC : CE.$$

Fig. 115 198. *Corollary* II. *If, between two straight lines* AB, CD (fig. 115), *parallels* AC, EF, GH, BD, *&c., be drawn, these two straight lines will be cut proportionally, and we shall have,*

$$AE : CF :: EG : FH :: GB : HD.$$

For, let O be the point of meeting of the straight lines, AB, CD; in the triangle OEF, the line AC being drawn parallel to the base EF, $OE : AE :: OF : CF$, or $OE : OF :: AE : CF$.

In the triangle OGH we have likewise

$$OE : EG :: OF : FH, \text{ or } OE : OF :: EG : FH;$$

therefore, on account of the common ratio $OE : OF$, these two proportions give

$$AE : CF :: EG : FH.$$

It may be demonstrated, in the same manner, that

$$EG : FH :: GB : HD,$$

and so on; therefore the lines AB, CD, are cut proportionally by the parallels EF, GH, &c.

THEOREM.

199. *Reciprocally, if the sides* AB, AC (fig. 116), *are cut proportionally by the line* DE, *so that* AD : DB : : AE : EC, *the line* DE *will be parallel to the base* BC. Fig. 116.

Demonstration. If DE is not parallel to BC, let us suppose that DO is parallel to it; then, according to the preceding theorem $AD : DB :: AO : OC$.

But, by hypothesis, $AD : DB :: AE : EC$;

consequently $AO : OC :: AE : EC$,

which is impossible, since of the antecedents AE is greater than AO, and of the consequents EC is less than OC; hence the line, drawn through the point D parallel to BC, does not differ from DE; therefore DE is this line.

200. *Scholium.* The same conclusion might be deduced from the proportion $AB : AD :: AC : AE$.

For this proportion would give (IV)

$$AB - AD : AD :: AC - AE : AE, \text{ or } BD : AD :: EC : AE.$$

THEOREM.

201. *The line* AD (fig. 117), *which bisects the angle* BAC *of a triangle, divides the base* BC *into two segments* BD, DC, *proportional to the adjacent sides* AB, AC; *so that*, BD : DC : : AB : AC. Fig. 117.

Demonstration. Through the point C draw CE parallel to AD to meet BA produced.

In the triangle BCE, the line AD being parallel to the base (196), CE,

$$BD : DC :: AB : AE.$$

But the triangle ACE is isosceles; for, on account of the parallels AD, CE, the angle $ACE = DAC$, and the angle $AEC = BAD$ (67); and, by hypothesis, $DAC = BAD$; therefore the angle $ACE = AEC$, and consequently $AE = AC$ (48); substituting then AC for AE in the preceding proportion, we have

$$BD : DC :: AB : AC.$$

THEOREM.

202. *Two equiangular triangles have their homologous sides proportional and are similar.*

Fig. 119. *Demonstration.* Let ABC, CDE (*fig.* 119), be two triangles, which have their angles equal, each to each, namely, $BAC = CDE$, $ABC = DCE$, and $ACB = DEC$; the homologous sides, or those adjacent to the equal angles, will be proportional, that is,

$$BC : CE :: BA : CD :: AC : DE.$$

Let the homologous sides BC, CE, be in the same straight line, and produce the sides BA, ED, till they meet in F.

Since BCE is a straight line, and the angle $BCA = CED$, it follows that AC is parallel to DE (67). Also, since the angle $ABC = DCE$, the line AB is parallel to DC; therefore the figure $ACDF$ is a parallelogram.

In the triangle BFE, the line AC being parallel to the base FE, $BC : CE :: BA : AF$ (196); substituting in the place of AF its equal CD, we have

$$BC : CE :: BA : CD.$$

In the same triangle BFE, BF being considered as the base, since CD is parallel to BF, $BC : CE :: FD : DE$. Substituting for FD its equal AC we have

$$BC : CE :: AC : DE.$$

From these two proportions, which contain the same ratio $BC : CE$, we have

$$AC : DE :: BA : CD.$$

Hence the equiangular triangles BAC, CDE, have the homologous sides proportional. But two figures are similar, when they have, at the same time, their angles equal, each to each, and the homologous sides proportional (162); therefore the equiangular triangles BAC, CDE, are two similar figures.

203. *Corollary.* In order to be similar, it is sufficient that two triangles have two angles of the one respectively equal to two angles of the other; for then the third angles will be equal and the two triangles will be equiangular.

204. *Scholium.* It may be remarked, that in similar triangles the homologous sides are opposite to equal angles; thus, the angle *ACB* being equal to *DEC*, the side *AB* is homologous to *DC*; likewise *AC*, *DE*, are homologous, being opposite to the equal angles *ABC*, *DCE*. Knowing the homologous sides, we readily form the proportions;

$$AB : DC :: AC : DE :: BC : CE.$$

THEOREM.

205. *Two triangles, which have their homologous sides proportional, are equiangular and similar.*

Demonstration. Let us suppose that

$$BC : EF :: AB : DE :: AC : DF \text{ (fig. 120)};$$ Fig. 120

the triangles *ABC*, *DEF*, will have their angles equal, namely, $A = D$, $B = E$, $C = F$.

Make, at the point *E*, the angle $FEG = B$, and at the point *F*, the angle $EFG = C$, the third angle *G* will be equal to the third angle *A*, and the two triangles *ABC*, *EFG*, will be equiangular; whence, by the preceding theorem, $BC : EF :: AB : EG$; but, by hypothesis, $BC : EF :: AB : DE$; consequently $EG = DE$. We have, moreover, by the same theorem, $BC : EF :: AC : FG$; but, by hypothesis, $BC : EF :: AC : DF$; consequently $FG = DF$; hence the triangles *EGF*, *DEF*, have the three sides of the one equal to the three sides of the other, each to each; they are therefore equal (43). But, by construction, the triangle *EGF* is equiangular with the triangle *ABC*; therefore the triangles *DEF*, *ABC*, are, in like manner, equiangular and similar.

206. *Scholium.* It will be perceived, by the two last propositions, that, when the angles of one triangle are respectively equal to those of another, the sides of the former are proportional to those of the latter, and the reverse; so that one of these conditions is sufficient to establish the similitude of triangles. This is not true of figures having more than three sides; for, with respect to those of only four sides, or quadrilaterals, we may alter the proportion of the sides without changing the angles, or change the angles without altering the sides; thus, from the angles being equal it does not follow that the sides are proportional, or the reverse. We see, for example, that by drawing *EF* (*fig.* 121) parallel to Fig 121 *BC*, the angles of the quadrilateral *AEFD* are equal to those of the

quadrilateral $ABCD$; but the proportion of the sides is different. Also, without changing the four sides AB, BC, CD, AD, we can bring the points B and D nearer together, or remove them further apart, which would alter the angles.

207. *Scholium.* The two preceding theorems (202, 205), which, properly speaking, make only one, added to that of the square of the hypothenuse (186), are of all the propositions of geometry the most remarkable for their importance and the number of results that are derived from them; they are almost sufficient of themselves, for all applications and for the resolution of all problems; the reason is, that all figures may be resolved into triangles, and any triangle whatever into two right-angled triangles. Thus the general properties of triangles involve those of all figures.

THEOREM.

208. *Two triangles, which have an angle of the one equal to an angle of the other and the sides about these angles proportional, are similar.*

f. 122. *Demonstration.* Let the angle $A = D$ (*fig.* 122), and let $AB : DE :: AC : DF$, the triangle ABC is similar to the triangle DEF.

Take $AG = DE$, and draw GH parallel to BC, the angle $AGH = ABC$ (67); and the triangle AGH will be equiangular with the triangle ABC;

whence $AB : AG :: AC : AH$;

but, by hypothesis, $AB : DE :: AC : DF$,

and, by construction, $AG = DE$; therefore $AH = DF$. The two triangles AGH, DEF, have the two sides and the included angle of the one respectively equal to two sides and the included angle of the other; they are consequently equal. But the triangle AGH is similar to ABC; therefore DEF is also similar to ABC.

THEOREM.

209. *Two triangles, which have the sides of the one parallel, or which have them perpendicular, to those of the other, each to each, are similar.*

f. 123. *Demonstration.* 1. If the side AB (*fig.* 123) is parallel to DE, and BC to EF, the angle ABC will be equal to DEF (70); if,

moreover, AC is parallel to DF, the angle ACB will be equal to DFE, and also BAC to EDF; therefore the triangles ABC, DEF, are equiangular and consequently similar.

2. Let the side DE (*fig.* 124) be perpendicular to AB, and the side DF to AC. In the quadrilateral $AIDH$ the two angles I, H, will be right angles, and the four angles will be together equal to four right angles (80); therefore the two remaining angles IAH, IDH, are together equal to two right angles. But the two angles EDF, IDH, are together equal to two right angles, consequently the angle EDF is equal to IAH or BAC. In like manner, if the third side EF is perpendicular to the third side BC, it may be shown that the angle $DFE = C$, and $DEF = B$; therefore the two triangles ABC, DEF, which have the sides of the one perpendicular to those of the other, each to each, are equiangular and similar. Fig. 1

210. *Scholium.* In the first of the above cases the homologous sides are the parallel sides, and in the second the homologous sides are those which are perpendicular to each other. Thus, in the second case, DE is homologous to AB, DF to AC, and EF to BC.

The case of the perpendicular sides admits of the two triangles being differently situated from those represented in figure 124; but the equality of the respective angles may always be proved, either by means of quadrilaterals, such as $AIDH$, which have two right angles, or by comparing two triangles which, beside the vertical angles, have each a right angle; or we can always suppose, within the triangle ABC, a triangle DEF, the sides of which shall be parallel to those of the triangle to be compared with ABC, and then the demonstration will be the same as that given for the case of figure 124.

THEOREM.

211. *Lines* AF, AG, *&c.* (fig. 125), *drawn at pleasure through the vertex of a triangle, divide proportionally the base* BC *and its parallel* DE, *so that* Fig.

$$DI : BF :: IK : FG :: KL : GH, \ \&c.$$

Demonstration. Since DI is parallel to BF, the triangles ADI, ABF, are equiangular, and $DI : BF :: AI : AF$; also, IK being parallel to FG, $AI : AF :: IK : FG$; hence, on account of

the common ratio, $AI : AF, DI : BF :: IK : FG$. It may be shown, in like manner, that $IK : FG :: KL : GH$, &c.; therefore the line DE is divided at the points I, K, L, as the base BC is at the points F, G, H.

212. *Corollary.* If BC should be divided into equal parts at the points F, G, H, the parallel DE would be divided likewise into equal parts at the points I, K, L.

THEOREM.

Fig. 126. 213. *If from the right angle* A (fig. 126) *of a right-angled triangle the perpendicular* AD *be let fall upon the hypothenuse;*

1. *The two partial triangles* ABD, ADC, *will be similar to each other and to the whole triangle* ABC;

2. *Each side* AB *or* AC *will be a mean proportional between the hypothenuse* BC *and the adjacent segment* BD *or* DC;

3. *The perpendicular* AD *will be a mean proportional between the two segments* BD, DC.

Demonstration. 1. The triangles BAD, BAC, have the angle B common; moreover the right angle $BDA = BAC$; consequently the third angle BAD of the one is equal to the third angle C of the other, and the two triangles are equiangular and similar. It may be demonstrated, in the same manner, that the triangle DAC is similar to the triangle BAC; therefore the three triangles are equiangular and similar.

2. Since the triangle BAD is similar to the triangle BAC, their homologous sides are proportional. Now the side BD in the smaller triangle is homologous to the side BA in the larger, because they are opposite to the equal angles, BAD, BCA; the hypothenuse BA of the smaller is homologous to the hypothenuse BC of the larger;

hence $$BD : BA :: BA : BC.$$

In the same manner it may be shown that

$$DC : AC :: AC : BC;$$

therefore each of the sides AB, AC, is a mean proportional between the hypothenuse and the segment adjacent to this side.

3. By comparing the homologous sides of the similar triangles ABD, ADC, we have

$$BD : AD :: AD : DC;$$

therefore the perpendicular AD is a mean proportional between the segments BD, DC, of the hypothenuse.

214. *Scholium.* The proportion $BD : AB :: AB : BC$, by putting the product of the extremes equal to that of the means, gives

$$\overline{AB}^2 = BD \times BC.$$

We have, in like manner,

$$\overline{AC}^2 = DC \times BC,$$

hence $\overline{AB}^2 + \overline{AC}^2 = BD \times BC + DC \times BC$;

the second member, otherwise expressed, is $(BD + DC) \times BC$, or $\overline{BC}^2$;

consequently $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$;

therefore the square of the hypothenuse BC is equal to the sum of the squares of the two other sides AB, AC. We thus fall again upon the proposition of the square of the hypothenuse by a process very different from that before pursued; from which it appears, that, properly speaking, the proposition of the square of the hypothenuse is a consequence of the proportionality of the sides of equiangular triangles. Thus the fundamental propositions of geometry reduce themselves, as it were, to this single one, that equiangular triangles have their homologous sides proportional.

It often happens, as in the present instance, that by pursuing the consequences of one or several propositions, we return to the propositions before demonstrated. Generally speaking, that which particularly characterizes the theorems of geometry, and which is an irresistible proof of their certainty, is, that by combining them together in any manner whatever, provided the reasoning be just, we always fall upon accurate results. This would not be the case, if any proposition were false, or only true to a certain degree; it would often happen, that, by combining the propositions together, the error would augment and become sensible. We have examples of this in all those demonstrations, in which we make use of the *reductio ad absurdum*. These demonstrations, in which the object is to prove that two quantities are equal, consist in making it evident, that, if there were between them the least inequality, we should be led by a course of reasoning to a manifest and palpable absurdity; whence we are obliged to conclude that the two quantities are equal.

Fig. 127. 215. *Corollary.* If from the point A (*fig.* 127) of the circumference of a circle two chords AB, AC, be drawn to the extremities of the diameter BC, the triangle ABC will be right-angled at A (128); whence, 1. *the perpendicular* AD *is a mean proportional between the segments* BD, DC, *of the diameter*, or, which amounts to the same thing,

$$\overline{AD}^2 = BD \times DC.$$

2. *The chord* AB *is a mean proportional between the diameter* BC *and the adjacent segment* BD;

or, $$\overline{AB}^2 = BD \times BC.$$

Also $\overline{AC}^2 = DC \times BC$; therefore $\overline{AB}^2 : \overline{AC}^2 :: BD : DC$. If we compare $\overline{AB}^2$ with $\overline{BC}^2$, we shall have

$$\overline{AB}^2 : \overline{BC}^2 :: BD : BC;$$

we have, in like manner,

$$\overline{AC}^2 : \overline{BC}^2 :: DC : BC.$$

These ratios of the squares of the sides to each other and to the square of the hypothenuse have already been given in articles 189, 190.

* THEOREM.

216. *Two triangles, which have an angle in the one equal to an angle in the other, are to each other as the rectangles of the sides*
Fig. 128. *which contain the equal angles; thus, the triangle* ABC (fig. 128) *is to the triangle* ADE, *as the rectangle* AB × AC *is to the rectangle* AD × AE.

Demonstration. Draw BE; the two triangles ABE, ADE, whose common vertex is E, have the same altitude, and are to each other as their bases AB, AD (177); hence

$$ABE : ADE :: AB : AD.$$

In like manner,

$$ABC : ABE :: AC : AE.$$

multiplying the two proportions in order and omitting the common term ABE, we have,

$$ABC : ADE :: AB \times AC : AD \times AE.$$

217. *Corollary.* The two triangles would be equivalent, if the rectangle $AB \times AC$ were equal to the rectangle $AD \times AE$, or if

$AB : AD :: AE : AC$, which is the case when the line DC is parallel to BE.

THEOREM.

218. *Two similar triangles are to each other as the squares of their homologous sides.*

Demonstration. Let the angle $A = D$ (*fig.* 122), and the angle $B = E$, then, by the preceding proposition, Fig. 122.

$$ABC : DEF :: AB \times AC : DE \times DF;$$

and, because the triangles are similar,

$$AB : DE :: AC : DF.$$

This proportion being multiplied in order by the identical proportion

$$AC : DF :: AC : DF,$$

we shall have

$$AB \times AC ; DE \times DF :: \overline{AC}^2 : \overline{DF}^2.$$

Hence

$$ABC : DEF :: \overline{AC}^2 : \overline{DF}^2.$$

Therefore two similar triangles ABC, DEF, are to each other as the squares of the homologous sides AC, DF, or as the squares of any other two homologous sides.

THEOREM.

219. *Two similar polygons are composed of the same number of triangles, which are similar to each other and similarly disposed.*

Demonstration. In the polygon $ABCDE$ (*fig.* 129) draw from an angle A the diagonals AC, AD, to the other angles. In the other polygon $FGHIK$ draw, in like manner, from the angle F, homologous to A, the diagonals FH, FI, to the other angles. Fig. 129.

Since the polygons are similar, the angle ABC is equal to the homologous angle FGH (162), moreover the sides AB, BC, are proportional to the sides FG, GH, so that

$$AB : FG :: BC : GH.$$

It follows from this, that the triangles ABC, FGH, having an angle of the one equal to an angle of the other and the sides about the equal angles proportional, are similar (208), consequently the angle $BCA = GHF$. These equal angles being subtracted from the equal angles BCD, GHI, the remaining

angles ACD, FHI, will be equal. Now, since the triangles ABC, FGH, are similar,

$$AC : FH :: BC : GH$$

besides, on account of the polygons being similar (162),

$$BC : GH :: CD : HI;$$

consequently $$AC : FH :: CD : HI;$$

but we have seen that the angle $ACD = FHI$; consequently the triangles ACD, FHI, have an angle of the one equal to an angle of the other and the sides about the equal angles proportional; they are therefore similar (208). We might proceed in the same manner to demonstrate, that the remaining triangles are similar, whatever be the number of the sides of the proposed polygons; therefore two similar polygons are composed of the same number of triangles, which are similar to each other and similarly disposed.

220. *Scholium.* The converse of this proposition is equally true; *if two polygons are composed of the same number of triangles, which are similar to each other and similarly disposed, these two polygons will be similar.*

For, the triangles being similar, the angles $ABC = FGH$, $BCA = GHF$, $ACD = FHI$; consequently $BCD = GHI$, also $CDE = HIK$, &c. Moreover,

$$AB : FG :: BC : GH :: AC : FH :: CD : HI, \text{ \&c.};$$

consequently the two polygons have their angles respectively equal and their sides proportional; therefore they are similar.

THEOREM.

221. *The perimeters of similar polygons are as their homologous sides, and their surfaces are as the squares of these sides.*

Demonstration. 1. By the nature of similar figures we have

Fig. 129. $$AB : FG :: BC : GH :: CD : HI, \text{ \&c. } (fig.\ 129),$$

and from this series of equal ratios we may infer, that the sum of the antecedents $AB + BC + CD$, &c., the perimeter of the first figure is to the sum of the consequents $FG + GH + HI$, &c., the perimeter of the second figure, as one antecedent is to its consequent (IV), or as the side AB is to its homologous side FG.

2. The triangles ABC, FGH, being similar

$$ABC : FGH :: \overline{AC}^2 : \overline{FH}^2 \quad (218);$$

in like manner, *ACD, FHI*, being similar,

$$ACD : FHI :: \overline{AC}^2 : \overline{FH}^2;$$

hence, on account of the common ratio $\overline{AC}^2 : \overline{FH}^2$,

$$ABC : FGH :: ACD : FHI.$$

By a similar process of reasoning it may be shown that

$$ACD : FHI :: ADE : FIK;$$

and so on, if there should be a greater number of triangles. Hence, from this series of equal ratios, the sum of the antecedents $ABC + ACD + ADE$, or the polygon *ABCDE*, is to the sum of consequents $FGH + FHI + FIK$, or the polygon *FGHIK*, as one antecedent *ABC* is to its consequent *FGH*, or as $\overline{AB}^2$ is to $\overline{FG}^2$ (219). Therefore the surfaces of similar polygons are to each other, as the squares of their homologous sides.

222. *Corollary.* If three similar figures be constructed whose homologous sides are equal to the three sides of a right-angled triangle, the figure described upon the greatest side will be equal to the sum of the two others; for the three figures will be proportional to the squares of their homologous sides; now the square of the hypothenuse is equal to the sum of the squares of the two other sides; therefore, &c.

THEOREM.

223. *The parts of two chords which cut each other in a circle are reciprocally proportional; that is,* AO : DO : : CO : OB (fig. 130). Fig. 13

Demonstration. Join *AC* and *BD*. In the triangles *ACO, BOD*, the angles at *O* are equal, being vertical angles, and the angle *A* is equal to the angle *D*, because they are inscribed in the same segment (127); for the same reason the angle $C = B$; therefore these triangles are similar, and the homologous sides give the proportion

$$AO : DO :: CO : OB.$$

224. *Corollary.* Hence $AO \times OB = DO \times CO$; therefore the rectangle of the two parts of one of the cords is equal to the rectangle of the two parts of the other.

THEOREM.

225. *If from a point* O (fig. 131), *taken without a circle, secants* OB, OC, *be drawn terminating in the concave arc* BC, the entire Fig. 1

secants will be reciprocally proportional to the parts without the circle; that is, OB : OC : : OD : OA.

Demonstration. Join AC and BD. The triangles OAC, OBD, have the angle O common; moreover the angle $B = C$ (126); therefore the triangles are similar; and the homologous sides give the proposition

$$OB : OC :: OD : OA.$$

226. *Corollary.* The rectangle $OA \times OB = OC \times OD$.

227. *Scholium.* It may be remarked, that this proposition has great analogy with the preceding; the only difference is, that the two chords AB, CD, instead of intersecting each other in the circle, meet without it. The following proposition may also be regarded as a particular case of this.

*

THEOREM.

Fig. 132. 228. *If from the same point* O (fig. 132), *taken without the circle, a tangent* OA *be drawn and a secant* OC, *the tangent will be a mean proportional between the secant and the part without the circle; that is,* OC : OA : : OA : OD, *or*, $\overline{OA}^2 = OC \times OD$.

Demonstration. By joining AD and AC, the triangles OAD, OAC, have the angle O common; moreover, the angle OAD formed by a tangent and a chord (131) has for its measure the half of the arc AD, and the angle C has the same measure; consequently the angle $OAD = C$; therefore the two triangles are similar, and $OC : OA :: OA : OD$, which gives $\overline{OA}^2 = OC \times OD$.

THEOREM.

Fig. 133. 229. *In any triangle* ABC (fig. 133), *if the angle* A *be bisected by the line* AD, *the r ctangle of the sides* AB, AC, *will be equal to the rectangle of the segments* BD, DC, *plus the square of the bisecting line* AD.

Demonstration. Describe a circle the circumference of which shall pass through the points A, B, C; produce AD till it meet the circumference, and join CE.

The triangle BAD is similar to the triangle EAC; for, by hypothesis, the angle $BAD = EAC$; moreover the angle $B = E$, since they have each for their measure the half of the arc AC;

consequently the triangles are similar; and the homologous sides give the proportion

$$BA : AE :: AD : AC;$$

whence $BA \times AC = AE \times AD$; but $AE = AD + DE$, and, by multiplying each by AD, we have $AE \times AD = \overline{AD}^2 + AD \times DE$; besides, $AD \times DE = BD \times DC$ (224); therefore

$$BA \times AC = \overline{AD}^2 + BD \times DC.$$

THEOREM.

230. *In every triangle* ABC (fig. 134) *the rectangle of two of* Fig. 134.
the sides AB, AC, *is equal to the rectangle contained by the diameter* CE *of the circumscribed circle and the perpendicular* AD, *let fall upon the third side* BC.

Demonstration. Join AE, and the triangles ABD, AEC, are right-angled, the one at D and the other at A; moreover the angle $B = E$; consequently the triangles are similar; and they give the proportion, $AB : CE :: AD : AC$; whence

$$AB \times AC = CE \times AD.$$

231. *Corollary.* If these equal quantities be multiplied by BC, we shall have $AB \times AC \times BC = CE \times AD \times BC$. Now $AD \times BC$ is double the surface of the triangle (176); therefore *the product of the three sides of a triangle is equal to the surface multiplied by double the diameter of the circumscribed circle.*

The product of three lines is sometimes called a *solid*, for a reason that will be given hereafter. The value of this product is easily conceived by supposing the three lines reduced to numbers and these numbers multiplied together.

232. *Scholium.* It may be demonstrated also, that *the surface of a triangle is equal to its perimeter multiplied by half of the radius of the inscribed circle.*

For the triangles AOB, BOC, AOC (*fig.* 87), which have their Fig. 87.
common vertex in O, have for their common altitude the radius of the inscribed circle; consequently the sum of these triangles will be equal to the sum of the bases AB, BC, AC, multiplied by half of the radius OD therefore the surface of the triangle ABC is equal to the product of its perimeter by half of the radius of the inscribed circle.

THEOREM.

Fig. 135. 233. *In every inscribed quadrilateral figure* ABCD (fig. 135), *the rectangle of the two diagonals* AC, BD, *is equal to the sum of the rectangles of the opposite sides; that is,*

$$AC \times BD = AB \times CD + AD \times BC.$$

Demonstration. Take the arc $CO = AD$, and draw BO meeting the diagonal AC in I.

The angle $ABD = CBI$, since one has for its measure half of the arc AD, and the other half of CO equal to AD. The angle $ADB = BCI$, because they are inscribed in the same segment AOB; consequently the triangle ABD is similar to the triangle IBC, and $AD : CI :: BD : BC$; whence

$$AD \times BC = CI \times BD.$$

Again, the triangle ABI is similar to the triangle BDC; for the arc AD being equal to CO, if we add to each of these OD we shall have the arc $AO = DC$; consequently the angle ABI is equal to DBC; moreover the angle $BAI = BDC$, because they are inscribed in the same segment; therefore the triangles ABI, BDC, are similar, and the homologous sides give the proportion $AB : BD :: AI : CD$; whence,

$$AB \times CD = AI \times BD.$$

Adding the two results above found, and observing that

$$AI \times BD + CI \times BD = (AI + CI) \times BD = AC \times BD,$$

we have

$$AD \times BC + AB \times CD = AC \times BD.$$

234. *Scholium.* We may demonstrate, in a similar manner, another theorem with respect to an inscribed quadrilateral figure.

The triangle ABD being similar to BIC, $BD : BC :: AB : BI$, whence

$$BI \times BD = BC \times AB.$$

If we join CO, the triangle ICO, similar ABI, is similar to BDC, and gives the proportion $BD : CO :: DC : OI$, whence we have $OI \times BD = CO \times DC$, or, CO being equal to AD,

$$OI \times BD = AD \times DC.$$

Adding these two results, and observing that $BI \times BD + OI \times BD$ reduces itself to $BO \times BD$, we obtain

$$BO \times BD = AB \times BC + AD \times DC.$$

If we had taken $BP = AD$, and had drawn CKP, we should have found by similar reasoning

$$CP \times CA = AB \times AD + BC \times CD.$$

But the arc BP being equal to CO, if we add to each BC, we shall have the arc $CBP = BCO$; consequently the chord CP is equal to the chord BO, and the rectangles $BO \times BD$ and $CP \times CA$, are to each other as BD is to CA; therefore

$$BD : CA :: AB \times BC + AD \times DC : AB \times AD + BC \times CD;$$

that is, *the two diagonals of an inscribed quadrilateral figure are to each other as the sums of the rectangles of the sides adjacent to their extremities.*

By means of these two theorems the diagonals may be found, when the sides are known.

THEOREM.

255. *Let* P (fig. 136) *be a given point within a circle in the* Fig. 136
radius AC, *and let there be taken a point* Q *without the circle in the same radius produced such that* CP : CA : : CA : CQ ; *if, from any point* M *of the circumference, straight lines* MP, MQ, *be drawn to the points* P *and* Q, *these straight lines will always be in the same ratio, and we shall have* MP : MQ : : AP : AQ.

Demonstration. By hypothesis, $CP : CA :: CA : CQ$; putting CM in the place of CA we shall have $CP : CM :: CM : CQ$; consequently the triangles CPM, CQM, having an angle of the one equal to an angle of the other and the sides about the equal angles proportional, are similar (208); therefore

$$MP : MQ :: CP : CM \text{ or } CA.$$

But the proportion

$$CP : CA :: CA : CQ$$

gives, by *division*,

$$CP : CA :: CA - CP : CQ - CA,$$

or $$CP : CA :: AP : AQ;$$

therefore $$MP : MQ :: AP : AQ.$$

Problems relating to the third section.

PROBLEM.

236. *To divide a given straight line into any number of equal parts, or into parts proportional to any given lines.*

Fig. 137. *Solution.* 1. Let it be proposed to divide the line AB (*fig.* 137) into five equal parts; through the extremity A draw the indefinite straight line AG, and take AC of any magnitude whatever, and apply it five times upon AG; through the last point of the division G draw GB, and through C draw CI parallel to GB; AI will be a fifth part of the line AB, and, by applying AI five times upon AB, the line AB will be divided into five equal parts.

For, since CI is parallel to GB, the sides AG, AB, are cut proportionally in C and I (196). But AC is a fifth part of AG, therefore AI is a fifth part of AB.

Fig. 138. 2. Let it be proposed to divide the line AB (*fig.* 138) into parts proportional to the given lines P, Q, R. Through the extremity A draw the indefinite straight line AG, and take $AC = P$, $CD = Q$, $DE = R$; join EB, and through the points C, D, draw CI, DK, parallel to EB; the line AB will be divided at I and K into parts proportional to the given lines P, Q, R.

For, on account of the parallels CI, DK, EB, the parts AI, IK, KB, are proportional to the parts AC, CD, DE (196); and, by construction, these are equal to the given lines P, Q, R.

PROBLEM.

237. *To find a fourth proportional to three given lines* **A, B, C**

Fig. 139. (fig. 139).

Solution. Draw the two indefinite lines DE, DF, making any angle with each other. On DE take $DA = A$, $DB = B$; and upon DF take $DC = C$; join AC, and through the point B draw BX parallel to AC; DX will be the fourth proportional required.

For, since BX is parallel to AC, $DA : DB :: DC : DX$; but the three first terms of this proportion are equal to the three given lines, therefore DX is the fourth proportional required.

238. *Corollary.* We might find in the same manner a third proportional to two given lines A, B; for it would be the same as the fourth proportional to the three lines A, B, B.

PROBLEM.

239. *To find a mean proportional between two given lines* A *and* B (fig. 140). Fig 140

Solution. On the indefinite line DF take $DE = A$, and $EF = B$; on the whole line DF, as a diameter, describe the semicircumference DGF; at the point E erect upon the diameter the perpendicular EG meeting the circumference in G; EG will be the mean proportional sought.

For the perpendicular GE, let fall from a point in the circumference upon the diameter, is a mean proportional between the two segments of the diameter DE, EF (215), and these two segments are equal to the two given lines A and B.

PROBLEM.

240. *To divide a given line* AB (*fig.* 141) *into two parts in* Fig. 141 *such a manner, that the greater shall be a mean proportional between the whole line and the other part.*

Solution. At the extremity B of the line AB erect the perpendicular BC equal to half of AB; from the point C, as a centre, and with the radius CB, describe a circle; draw AC cutting the circumference in D, and take $AF = AD$; the line AB will be divided at the point F in the manner required; that is,

$$AB : AF :: AF : FB.$$

For AB, being a perpendicular to the radius CB at its extremity CB, is a tangent; and, if AC be produced till it meet the circumference in E, we shall have

$$AE : AB :: AB : AD \quad (228),$$

and hence $\quad AE - AB : AB :: AB - AD : AD \quad$ (IV).

But, since the radius BC is half of AB, the diameter DE is equal to AB, and consequently $AE - AB = AD = AF$; also, since $AF = AD$, $AB - AD = FB$; therefore,

$$AF : AB :: FB : AD \text{ or } AF,$$

and by *inversion* $\quad AB : AF :: AF : FB.$

241. *Scholium.* When a line is divided in this manner, it is said to be divided *in extreme and mean ratio.* Its application will be seen hereafter.

It may be remarked, that the secant AE is divided in extreme and mean ratio at the point D; for, since $AB = DE$,

$$AE : DE :: DE : AD.$$

PROBLEM.

Fig. 142. 242. *Through a given point* A (fig. 142) *in a given angle* BCD, *to draw a line* BD *in such a manner that the parts* AB, AD, *comprehended between the point* A *and the two sides of the angle shall be equal.*

Solution. Through the point A draw AE parallel to CD, take $BE = CE$, and through the points B and A draw BAD, which will be the line required.

For, AE being parallel to CD, $BE : EC :: BA : AD$, but $BE = EC$; therefore $BA = AD$.

PROBLEM.

243. *To make a square equivalent to a given parallelogram, or to a given triangle.*

Fig. 143. *Solution.* 1. Let $ABCD$ (*fig.* 143) be the given parallelogram, AB its base, and DE its altitude; between AB and DE find a mean proportional XY (239); the square described upon XY will be equivalent to the parallelogram $ABCD$.

For, by construction, $AB : XY :: XY : DE$; hence

$$\overline{XY}^2 = AB \times DE;$$

but $AB \times DE$ is the measure of the parallelogram, and $\overline{XY}^2$ is that of the square, therefore they are equivalent.

Fig. 144. 2. Let ABC (*fig.* 144) be the given triangle, BC its base, and AD its altitude; find a mean proportional between BC and half of AD, and let XY be this mean proportional; the square described upon XY will be equivalent to the triangle ABC.

For, since $BC : XY :: XY : \frac{1}{2} AD$, $\overline{XY}^2 = BC \times \frac{1}{2} AD$; therefore the square described upon XY is equivalent to the triangle ABC.

PROBLEM.

Fig. 145. 244. *Upon a given line* AD (fig. 145) *to construct a rectangle* ADEX *equivalent to a given rectangle* ABFC.

Solution. Find a fourth proportional to the three lines AD, AB, AC (127), and let AX be this fourth proportional; the rectangle contained by AD and AX will be equivalent to the rectangle $ABFC$.

For, since $AD : AB :: AC : AX$, $AD \times AX = AB \times AC$; therefore the rectangle $ADEX$ is equivalent to the rectangle $ABFC$.

PROBLEM.

245. *To find in lines the ratio of the rectangle of two given lines* A *and* B (fig. 148) *to the rectangle of two given lines* C *and* D. Fig. 148.

Solution. Let X be a fourth proportional to the three given lines B, C, D; the ratio of the two lines A and X will be equal to that of the two rectangles $A \times B$, $C \times D$.

For, since $B : C :: D : X$, $C \times D = B \times X$; therefore

$$A \times B : C \times D :: A \times B : B \times X :: A : X.$$

246. *Corollary.* In order to obtain the ratio of the squares described upon two lines A and C, find a third proportional X to the lines A and C, so that we may have the proportion

$$A : C :: C : X;$$

then we shall have $A^2 : C^2 :: A : X$.

PROBLEM.

247. *To find in lines the ratio of the product of three given lines* A, B, C (fig. 149), *to the product of three given lines* P, Q. R. Fig 149.

Solution. Find a fourth proportional X to the three given lines P, A, B; and a fourth proportional Y to the three given lines C, Q, R. The two lines X and Y will be to each other as the products $A \times B \times C$, $P \times Q \times R$.

For, since $P : A :: B : X$, $A \times B = P \times X$; and, by multiplying each of these by C, we shall have

$$A \times B \times C = C \times P \times X.$$

In like manner, since

$$C : Q :: R : Y,\ \ Q \times R = C \times Y;$$

and, by multiplying each of these by P, we shall have

$$P \times Q \times R = P \times C \times Y;$$

therefore the product

$A \times B \times C$: the product $P \times Q \times R :: C \times P \times X : P \times C \times Y :: X : Y$.

PROBLEM.

248. *To make a triangle equivalent to a given polygon.*

Solution. Let $ABCDE$ (*fig.* 146) be the given polygon. Fig. 146
Draw the diagonal CE, which cuts off the triangle CDE; through the point D draw DF parallel to CE to meet AE

produced; join CF, and the polygon $ABCDE$ will be equivalent to the polygon $ABCF$, which has one side less.

For the triangles CDE, CFE, have the common base CE, they are also of the same altitude, for their vertices D, F, are in a line DF parallel to the base; therefore the triangles are equivalent. Adding to each of these the figure $ABCE$ and we shall have the polygon $ABCDE$ equivalent to the polygon $ABCF$.

We can in like manner cut off the angle B by substituting for the triangle ABC the equivalent triangle AGC, and then the pentagon $ABCDE$ will be transformed into an equivalent triangle GCF.

The same process may be applied to any other figure; for, by making the number of sides one less at each step, we shall at length arrive at an equivalent triangle.

249. *Scholium.* As we have already seen, that a triangle may be transformed into an equivalent square (243), we may accordingly find a square equivalent to any given rectilineal figure; this is called *squaring* the rectilineal figure, or finding the *quadrature* of it.

The problem of the *quadrature of the circle* consists in finding a square equivalent to a circle whose diameter is given.

PROBLEM.

250. *To make a square which shall be equal to the sum or the difference of two given squares.*

Fig. 147. *Solution.* Let A and B (*fig.* 147) be the sides of the given squares.

1. If it is proposed to find a square equal to the sum of these squares, draw the two indefinite lines ED, EF, at right angles to each other; take $ED = A$ and $EG = B$; join DG, and DG will be the side of the square sought.

For the triangle DEG being right-angled, the square described upon DG will be equal to the sum of the squares described upon ED and EG.

2. If it is proposed to find a square equal to the difference of the given squares, form in like manner a right angle FEH, take GE equal to the smaller of the sides A and B; from the point G, as a centre, and with a radius GH equal to the other side, describe an arc cutting EH in H; the square described upon EH

will be equal to the difference of the squares described upon the lines A and B.

For in the right-angled triangle GEH the hypothenuse $GH = A$, and the side $GE = B$; therefore the square described upon EH is equal to the difference of the squares described upon the given sides A and B.

251. *Scholium.* We can thus find a square equal to the sum of any number of squares; for the construction by which two are reduced to one, may be used to reduce three to two and these two to one, and so of a larger number. Also a similar method may be employed when certain given squares are to be subtracted from others.

PROBLEM.

252. *To construct a square which shall be to a given square* ABCD (fig. 150), *as the line* M *is to the line* N. Fig. 150

Solution. On the indefinite line EG take $EF = M$, and $FG = N$; on EG, as a diameter, describe a semicircle, and at the point F erect upon the diameter the perpendicular FH. From the point H draw the chords HG, HE, which produce indefinitely; on the first take HK equal to the side AB of the given square, and through the point K draw KI parallel to EG; HI will be the side of the square sought.

For, on account of the parallels KI, GE,

$$HI : HK :: HE : HG;$$

hence $$\overline{HI}^2 : \overline{HK}^2 :: \overline{HE}^2 : \overline{HG}^2 \quad (\text{v}).$$

But, in the right-angled triangle EHG,

$$\overline{HE}^2 : \overline{HG}^2 :: \text{segment } EF : \text{the segment } FG \quad (215),$$

or, as M is to N;

therefore $$\overline{HI}^2 : \overline{HK}^2 :: M : N.$$

But $HK = AB$; therefore

the square upon HI : the square upon AB :: M : N.

PROBLEM.

253. *Upon a side* FG (fig. 129), *homologous to* AB, *to describe a polygon similar to a given polygon* ABCDE. Fig. 129

Solution. In the given polygon draw the diagonals AC, AD. At the point F make the angle $GFH = BAC$, and at the point G the angle $FGH = ABC$; the lines FH, GH, will cut each in H, and the triangle FGH will be similar to ABC. Likewise upon FH, homologous to AC, construct the triangle FIH similar to ADC, and upon FI, homologous to AD, construct the triangle FIK similar to ADE. The polygon $FGHIK$ will be similar to $ABCDE$.

For these two polygons are composed of the same number of triangles, which are similar to each other and similarly disposed (219).

PROBLEM.

254. *Two similar figures being given, to construct a similar figure which shall be equal to their sum or their difference.*

Solution. Let A and B be two homologous sides of the given figures, find a square equal to the sum or the difference of the squares described upon A and B; let X be the side of this square, X will be, in the figure sought, the side homologous to A and B in the given figures. The figure may then be constructed by the preceding problem.

For similar figures are as the square of their homologous sides; but the square of the side X is equal to the sum or the difference of the squares described upon the homologous sides A and B; therefore the figure described upon the side X is equal to the sum or the difference of the similar figures described upon the sides A and B.

PROBLEM.

255. *To construct a figure similar to a given figure, and which shall be to this figure in the given ratio of* M *to* N.

Solution. Let A be a side of the given figure, and X the homologous side of the figure sought; the square of X must be to the square of A, as M is to N (221); X then may be found by art. 251; and, knowing X, we may finish the problem by art. 252.

PROBLEM.

Fig. 151. 256. *To construct a figure similar to the figure* P (fig. 151) *and equivalent to the figure* Q.

Solution. Find the side M of a square equivalent to the figure P, and the side N of a square equivalent to the figure Q. Then let X be a fourth proportional to the three given lines M, N, AB; upon the side X, homologous to AB, describe a figure similar to the figure P; it will be equivalent to the figure Q.

For, by calling Y the figure described upon the side X, we shall have

$$P : Y :: \overline{AB}^2 : X^2.$$

But, by construction,

$$AB : X :: M : N,$$

or $$\overline{AB}^2 : X^2 :: M^2 : N^2;$$

therefore $$P : Y :: M^2 : N^2.$$

We have also, by construction, $M^2 = P$, and $N^2 = Q$;

consequently $$P : Y :: P : Q;$$

hence $Y = Q$; therefore the figure Y is similar to the figure P and equivalent to the figure Q.

PROBLEM.

257. *To construct a rectangle equivalent to a given square* C (fig. 152), *and whose adjacent sides shall make a given sum* AB. Fig 152

Solution. On AB, as a diameter, describe a semicircle, and draw DE parallel to the diameter, and at a distance AD, equal to a side of the given square C. From the point E, in which the parallel cuts the circumference, let fall upon the diameter the perpendicular EF; AF and FB will be the sides of the rectangle sought.

For their sum is equal to AB, and their rectangle $AF \times FB$ is equal to the square of EF (215), or of AD; therefore the rectangle is equivalent to the given square C.

258. *Scholium.* It is necessary in order that the problem may be possible, that the distance AD should not exceed the radius, that is, that the side of the square should not exceed half of the line AB.

PROBLEM.

259. *To construct a rectangle equivalent to a square* C (fig. 153), *and whose adjacent sides shall differ by a given quantity* AB. Fig. 153

Solution. On the given line AB, as a diameter, describe a circle; from the extremity of the diameter draw the tangent AD equal to the side of the square C. Through the point D and the centre O draw the secant DF; DE and DF will be the adjacent sides of the rectangle required.

For, 1. the difference of the sides is equal to the diameter EF or AB; 2. the rectangle $DE \times DF$ is equal to $\overline{AD}^2$ (228); therefore this rectangle will be equivalent to the given square C.

PROBLEM.

260. *To find the common measure, if there be one, between the diagonal and side of a square.*

Fig. 154. *Solution.* Let $ABCG$ (*fig.* 154) be any square and AC its diagonal.

We are, in the first place, to apply CB to CA, as often as it can be done (157); and in order to this, let there be described, from the centre C, and with a radius CB, the semicircle DBE. It will be seen, that CB is contained once in AC with a remainder AD; the result of the first operation therefore is the quotient 1 with the remainder AD, which is to be compared with BC, or its equal AB.

We may take $AF = AD$, and apply AF actually to AB; and we should find that it is contained twice with a remainder. But, as this remainder and the following ones go on diminishing and would soon become too small to be perceived, on account of the imperfection of the mechanical operation, from which we can conclude nothing with regard to the question, whether the lines AC, CB, have a common measure or not. Now there is a very simple method, by which we may avoid these decreasing lines, and which only requires an operation to be performed upon lines of the same magnitude.

The angle ABC being a right angle, AB is a tangent, and AE is a secant, drawn from the same point, so that

$$AD : AB :: AB : AE \quad (228).$$

Thus, in the second operation which has for its object to compare AD with AB, we may, instead of the ratio of AD to AB, take that of AB to AE. Now AB, or its equal CD, is contained twice in AE with a remainder AD; therefore the result of the second operation is the quotient 2 with the remainder AD, which is to be compared with AB.

The third operation, which consists in comparing AD with AB, reduces itself likewise to comparing AB, or its equal CD, with AE, and we have still the quotient 2 with the remainder AD.

Whence it is evident, that the operation will never terminate, and that accordingly there is no common measure between the diagonal and the side of a square, a truth already made known by a numerical operation, since these two lines are to each other $: : \sqrt{2} : 1$ (188), but which is rendered clearer by the geometrical solution.

261. It is not then possible to find in numbers the exact ratio of the diagonal to the side of a square; but we may approximate it to any degree we please by means of the continued fraction which is equal to this ratio. The first operation gave for a quotient 1; the second and each of the others continued without end gives 2; thus the fraction under consideration becomes

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\&c.}}}}}$$

If, for example, we take the four first terms of this fraction, we find that its value added to the first quotient 1 is $1\frac{12}{29}$, or $\frac{41}{29}$; so that the approximate ratio of the diagonal to the side of a square is $: : 41 : 29$. The ratio might be found more exactly by taking a greater number of terms.

SECTION FOURTH.

Of regular polygons and the measure of the circle.

DEFINITION.

262. A POLYGON, which is at the same time equiangular and equilateral, is called a *regular polygon.*

Regular polygons admit of any number of sides. The equilateral triangle is one of three sides; and the square one of four.

THEOREM.

263. *Two regular polygons of the same number of sides are similar figures.*

Demonstration. Let there be, for example, the two regular hexagons $ABCDEF$, $abcdef$ (*fig.* 155); the sum of the angles is the Fig. 155.

same in both, and is equal to eight right angles (82). The angle A is the sixth part of this sum as well as the angle a; therefore the two angles A and a are equal; the same may be said of the angles B and b, C and c, &c.

Moreover, since by the nature of these polygons the sides AB, BC, CD, &c., are equal, as also ab, bc, cd, &c.,

$$AB : ab :: BC : bc :: CD : cd, \text{ \&c.};$$

consequently the two figures under consideration have their angles equal and their homologous sides proportional; therefore they are similar (162).

264. *Corollary.* The perimeters of two regular polygons of the same number of sides are to each other as their homologous sides, and their surfaces are as the squares of these sides (221).

265. *Scholium.* The angle of a regular polygon is determined by the number of its sides like the angle of an equiangular polygon (79).

THEOREM.

266. *Every regular polygon may be inscribed in a circle and may be circumscribed about a circle.*

Fig. 156. *Demonstration.* Let $ABCDE$ &c. (*fig.* 156), be any regular polygon, and let there be described a circle, whose circumference shall pass through the three points A, B, C; let O be its centre, and OP a perpendicular let fall upon the middle of the side BC; join AO and OD.

The quadrilateral $OPCD$ may be placed upon the quadrilateral $OPBA$; in fact the side OP is common, and the angle $OPC = OPB$, each being a right angle, consequently the side PC will fall upon its equal PB, and the point C upon B. Moreover, by the nature of the polygon, the angle $PCD = PBA$; therefore CD will take the direction BA, and CD being equal to BA, the point D will fall upon A, and the two quadrilaterals will coincide throughout. Hence the distance OD is equal to the distance OA, and the circumference, which passes through the three points A, B, C, will pass also through the point D. By similar reasoning it may be shown, that the circumference, which passes through the three vertices B, C, D, will pass through the next vertex E, and so on; therefore the same circumference, which passes through the three points A, B, C,

passes through all the vertices of the angles of the polygon, and the polygon is inscribed in this circumference.

Furthermore, with respect to this circumference, all the sides AB, BC, CD, &c. are equal chords; they are accordingly equally distant from the centre (109); if therefore from the point O, as a centre, and with the radius OP, a circle be described, the circumference will touch the side BC and all the other sides of the polygon, each at its middle point, and the circle will be inscribed in the polygon, or the polygon will be circumscribed about the circle.

267. *Scholium* I. The point O, the common centre of the inscribed and circumscribed circle, may be regarded also as the centre of the polygon; and for this reason we call the *angle of the centre* the angle AOB formed by the two radii drawn to the extremities of the same side AB.

Since all the chords AB, BC, &c., are equal, it is evident that all the angles at the centre are equal, and that the value of each is found by dividing four right angles by the number of the sides of the polygon.

268. *Scholium* II. In order to inscribe a regular polygon of a certain number of sides in a given circle, it is only necessary to divide the circumference into as many equal parts as the polygon has sides; for, the arcs being equal, the chords AB, BC, CD, &c. (*fig.* 158), will be equal; the triangles ABO, BOC, COD, &c., Fig. 158 will also be equal, for the sides of the one will be respectively equal to those of the other; consequently all the angles ABC, BCD, CDE, &c., will be equal; therefore the figure $ABCDE$ &c. will be a regular polygon.

PROBLEM.

269. *To inscribe a square in a given circle.*

Solution. Draw the diameters AC, BD (*fig.* 157), cutting Fig. 157 each other at right angles; join the extremities A, B, C, D, and the figure $ABCD$ will be the inscribed square.

For, the angles AOB, BOC, &c., being equal, the chords AB, BC, &c., are equal.

270. *Scholium.* The triangle BOC being right-angled and isosceles, $BC : BO :: \sqrt{2} : 1$ (188); therefore, *the side of an inscribed square is to radius, as the square root of two is to unity.*

PROBLEM.

271. *To inscribe a regular hexagon and an equilateral triangle in a given circle.*

Solution. Let us suppose the problem resolved, and that AB
g. 158. (*fig.* 158) is a side of the inscribed hexagon; if we draw the radii AO, OB, the triangle AOB will be equilateral.

For the angle AOB is the sixth part of four right angles; thus, if we consider the right angle as unity, we shall have

$$AOB = \tfrac{4}{6} = \tfrac{2}{3}.$$

The two other angles ABO, BAO, of the same triangle, taken together, $= 2 - \frac{2}{3} = \frac{4}{3}$; and, as they are equal to each other, each of them $= \frac{2}{3}$; hence the triangle ABO is equilateral; therefore the side of the inscribed hexagon is equal to radius.

It follows from this, that in order to inscribe a regular hexagon in a given circle, the radius is to be applied six times round on the circumference, which will bring it to the point, from which the operation commenced.

The hexagon $ABCDEF$ being inscribed, if the vertices of the alternate angles A, C, E, be joined, an equilateral triangle ACE will be formed.

272. *Scholium.* The figure $ABCO$ is a parallelogram, and a rhombus, since $AB = BC = CO = AO$; therefore, the sum of the squares of the diagonals being equal to the sum of the squares of the sides (195),

$$\overline{AC}^2 + \overline{BO}^2 = 4\overline{AB}^2 = 4\overline{BO}^2;$$

subtracting $\overline{BO}^2$ from each we shall have

$$\overline{AC}^2 = 3\overline{BO}^2$$

hence $\overline{AC}^2 : \overline{BO}^2 :: 3 : 1$

or $AC : BO :: \sqrt{3} : 1$;

therefore *the side of an inscribed equilateral triangle is to radius as the square root of three is to unity.*

PROBLEM.

273. *To inscribe in a given circle a regular decagon, also a pentagon and a regular polygon of fifteen sides.*

g. 159. *Solution.* Divide the radius AO (*fig.* 159) in extreme and mean ratio at the point M (240), take the chord AB equal to the

greater segment OM, and AB will be the side of a regular decagon, which is to be applied ten times round on the circumference.

For, by joining MB, we have, by contruction,

$$AO : OM :: OM : AM,$$

or, because $$AB = OM,$$

$$AO : AB :: AB : AM;$$

therefore the triangles ABO, AMB, having an angle A common and the sides about this angle proportional, are similar (208). The triangle OAB is isosceles; consequently the triangle AMB is also isosceles, and $AB = BM$. Besides, $AB = OM$; hence also $BM = OM$; therefore the triangle BMO is isosceles.

The angle AMB, the exterior angle of the isosceles triangle BMO, is double of the interior angle O (78). Now the angle

$$AMB = MAB;$$

consequently the triangle OAB is such that each of the angles at the base OAB, OBA, is double of the angle at the vertex O, and the three angles of the triangle are equal to five times the angle O, and thus the angle O is a fifth part of two right angles, or the tenth part of four right angles; therefore the arc AB is the tenth part of the circumference, and the chord AB is the side of a regular decagon.

274. *Corollary* I. If the alternate vertices A, C, E, &c. of the decagon be joined, a regular pentagon $ACEGI$ will be formed.

275. *Corollary* II. AB being always the side of a decagon, let AL be the side of a hexagon; then the arc BL will be, with respect to the circumference, $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$; therefore the chord BL will be the side of a regular polygon of 15 sides. It is manifest, at the same time, that the arc CL is a third of CB.

276. *Scholium.* A regular polygon being formed, if the arcs subtended by the sides be bisected and chords to these half arcs be drawn, a regular polygon will be formed of double the number of sides. Thus by means of the square we may inscribe successively regular polygons of 8, 16, 32, &c., sides. Likewise by means of the hexagon we may inscribe regular polygons of 12, 24, 48, &c., sides; with the decagon, polygons of 20, 40, 80, &c., sides; with the regular polygon of fifteen sides, polygons of 30, 60, 120, &c., sides.*

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* It was supposed, for a long time, that these were the only polygons which could be inscribed by the processes of elementary geom-

PROBLEM.

Fig. 160. 277. *A regular inscribed polygon* ABCD &c. (fig. 160) *being given, to circumscribe about the same circle a similar polygon.*

Solution. At the point T, the middle of the arc AB, draw the tangent GH, which will be parallel to AB (112); do the same with each of the other arcs BC, CD, &c.; these tangents will form, by their intersections, the regular circumscribed polygon $GHIK$ &c., similar to the inscribed polygon.

It will be readily perceived, in the first place, that the three points O, B, H, are in a right line, for the right-angled triangles OTH, OHN, have the common hypothenuse OH, and the side $OT = ON$; they are consequently equal (126), and the angle $TOH = HON$, and the line OH passes through the point B, the middle of the arc TN. For the same reason, the point I is in OC produced, &c. But, since GH is parallel to AB, and HI to BC, the angle $GHI = ABC$ (67); in like manner $HIK = BCD$, &c.; hence the angles of the circumscribed polygon are equal to those of the inscribed polygon. Moreover, on account of these same parallels

$$GH : AB :: OH : OB,$$

and $$HI : BC :: OH : OB,$$

hence $$GH : AB :: HI : BC.$$

But $AB = BC$; consequently $GH = HI$. For the same reason $HI = IK$, &c.; consequently the sides of the circumscribed polygon are equal to each other; therefore this polygon is regular and similar to the inscribed polygon.

278. *Corollary* I. Reciprocally, if the circumscribed polygon $GHIK$ &c., be given, and it is proposed to construct, by means of it, the inscribed polygon $ABCD$ &c., it is evidently sufficient to draw to the vertices G, H, I, &c., of the given polygon the lines OG, OH, OI, &c., which will meet the circumference at the points A, B, C, &c., and then to join these points by the

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etry, or, which amounts to the same thing, by the resolution of equations of the first and second degree. But M. Gaus has shown in a work entitled *Disquisitiones Arithmeticæ*, Lipsiæ, 1801, that we may, by similar methods, inscribe a regular polygon of seventeen sides and in general one of $2^n + 1$ sides, provided that $2^n + 1$ be a prime number.

chords AB, BC, CD, &c., which will form the inscribed polygon. We might also, in this case, simply join the points of contact, T, N, P, &c., by the chords TN, NP, PQ, &c., which would equally form an inscribed polygon similar to the circumscribed one.

279. *Corollary* II. There may be circumscribed, about a given regular circle, all the polygons which can be inscribed within it; and, reciprocally, there may be inscribed, within a circle, all the polygons that can be circumscribed about it.

THEOREM.

280. *The area of a regular polygon is equal to the product of its perimeter by half of the radius of the inscribed circle.*

Demonstration. Let there be, for example, the regular polygon $GHIK$ &c. (*fig.* 160); the triangle GOH, for example, has for its measure $GH \times \frac{1}{2}OT$, the triangle OHI has for its measure $HI \times \frac{1}{2}ON$. But $ON = OT$; consequently the two triangles united have for their measure $(GH + HI) + \frac{1}{2}OT$. By proceeding thus with the other triangles, it is evident that the sum of all the triangles, or the entire polygon, has for its measure the sum of the bases GH, HI, IK, &c., or the perimeter of the polygon, multiplied by $\frac{1}{2}OT$, half of the radius of the inscribed circle. Fig. 160

281. *Scholium.* The radius of the inscribed circle is the same as the perpendicular let fall from the centre upon one of the sides.

THEOREM.

282. *The perimeters of regular polygons of the same number of sides are as the radii of the circumscribed circles, and also as the radii of the inscribed circles; and their surfaces are as the squares of these same radii.*

Demonstration. Let AB (*fig.* 161) be a side of one of the polygons in question, O its centre, and OA the radius of the circumscribed circle, and OD, perpendicular to AB, the radius of the inscribed circle; and let ab be the side of another polygon, similar to the former, o its centre, oa and od the radii of the circumscribed and inscribed circles. Fig. 161

The perimeters of the two polygons are to each other as the sides AB, ab (221). Now the angles A and a are equal, being each half of the angle of the polygon; the same may be said of

the angles B and b; therefore the triangles ABO, abo, are similar, as also the right-angled triangles ADO, ado;

hence $$AB : ab :: AO : ao :: DO : do;$$

consequently the perimeters of the polygons are to each other as the radii AO, ao, of the circumscribed circles, and also as the radii DO, do, of the inscribed circles.

The surfaces of these same polygons are to each other as the squares of the homologous sides AB, ab (221); they are therefore also as the squares of the radii of the circumscribed circles AO, ao, and as the squares of the radii of the inscribed circles DO, do.

LEMMA.

283. *Every curved line, or polygon, which encloses, from one*
Fig. 162. *extremity to the other, a convex line* AMB (fig. 162), *is greater than the enclosed line* AMB.

Demonstration. We have already said that, by a convex line, we understand a curved line or polygon, or a line consisting in part of a curve and in part of a polygon, such that a straight line cannot cut it in more than two points (83). If the line AMB had reentering parts or sinuosities, it would cease to be convex, because, as will be readily perceived, it might be cut by a straight line in more than two points. The arcs of a circle are essentially convex; but the proposition under consideration extends to every line, which fulfils the condition stated.

This being premised, if the line AMB be not smaller than any of those lines which enclose it, there is among these last a line smaller than any of the others, which is less than AMB, or at least equal to AMB. Let $ACDEB$ be this enclosing line; between these two lines draw at pleasure the straight line PQ, which does not meet the line AMB, or at most only touches it; the straight line PQ is less than $PCDEQ$ (3); consequently if, instead of $PCDEQ$, we substitute the straight line PQ, we shall have the enclosing line $APQB$, less than $APDQB$. But, by hypothesis, this must be the shortest of all; this hypothesis then cannot be maintained; therefore each of the enclosing lines is greater than AMB.

284. *Scholium.* After the same manner, it may be demonstrated, without any restriction, that a line which is convex and

returns into itself, *AMB* (*fig.* 163), is less than any line which Fig. 16
encloses it on all sides, whether the enclosing line *FHG* touches *AMB* in one, or more points, or whether it surrounds it without touching it.

LEMMA.

285. *Two concentric circles being given, there may always be inscribed, in the greater, a regular polygon, the sides of which shall not meet the circumference of the smaller; and there may also be circumscribed, about the smaller, a regular polygon, the sides of which shall not meet the circumference of the greater; so that on the whole the sides of the polygon described shall be contained between the two circumferences.*

Demonstration. Let *CA*, *CB* (*fig.* 164), be the radii of the two Fig. 16
given circles. At the point *A* draw the tangent *DE* terminating, at the greater circumference, in *D* and *E*. Inscribe, in the greater circumference, one of the regular polygons, which can be inscribed by the preceding problems, and bisect the arcs subtended by the sides, and draw the chords of these half arcs; and a regular polygon will be described of double the number of sides. Continue to bisect the arcs until one is obtained which is smaller than *DBE*. Let *MBN* be this arc, the middle of which is supposed to be in *B*; it is evident that the chord *MN* will be further from the centre than *DE*, and that thus the regular polygon, of which *MN* is a side, cannot meet the circumference, of which *CA* is the radius.

The same things being supposed, join *CM* and *CN*, which meet the tangent *DE* in *P* and *Q*; *PQ* will be the side of a polygon circumscribed about the smaller circumference similar to the polygon inscribed in the greater, the side of which is *MN*. Now it is evident that the circumscribed polygon, which has for its side *PQ*, cannot meet the greater circumference, since *CP* is less than *CM*.

There may, therefore, by the same construction, be a regular polygon inscribed in the greater circumference, and a similar polygon circumscribed about the smaller, which shall have their sides comprehended between the two circumferences.

286. *Scholium.* If we have two concentric sectors *FCG*, *ICH*, we can likewise inscribe, in the greater, *a portion of a regular polygon*, or circumscribe, about the smaller, a portion of a similar

polygon, so that the perimeters of the two polygons would be comprehended between the two circles. It is only necessary to divide the arc *FBG* successively into 2, 4, 8, 16, &c., equal parts, until one is obtained smaller than *DBE*.

By a *portion of a regular polygon*, as the phrase is here used, is to be understood the figure terminated by a series of equal chords, inscribed in the arc *FG*, from one extremity to the other. This portion has the principal properties of a regular polygon, it has its angles equal, and its sides equal; it is, at the same time, capable of being inscribed in, and circumscribed about, a circle; it does not, however, make a part of a regular polygon, properly so called, except when the arc, subtended by one of these sides, is an aliquot part of the circumference.

THEOREM.

287. *The circumferences of circles are as their radii, and their surfaces are as the squares of their radii.*

Fig. 165. *Demonstration.* Denoting, by *circ. CA* and *circ. OB* (*fig.* 165), the circumferences of the circles whose radii are *CA* and *OB*, we say that *circ. CA* : *circ. OB* : : *CA* : *OB*.

For, if this proportion be not true, *CA* will be to *OB* as *circ. CA* is to a fourth term either greater or less than *circ. OB*. Let us suppose that it is less, and that, if possible,

$$CA : OB :: circ.\ CA : circ.\ OD.$$

Inscribe, in the circumference of which *OB* is the radius, a regular polygon *EFGKLE*, whose sides shall not meet the circumference of the circle, whose radius is *OD* (285); inscribe a similar polygon *MNPSTM* in the circle whose radius is *CA*.

This being done, since the polygons are similar, their perimeters *MNPSM*, *EFGKE*, are to each other as the radii *CA*, *OB*, of the circumscribed circles (282), and we have

$$MNPSM : EFGKE :: CA : OB;$$

but, by hypothesis,

$$CA : OB :: circ.\ CA : circ.\ OD;$$

therefore *MNPSM* : *EFGKE* : : *circ. CA* : *circ. OD*. Now this proportion is impossible, because the perimeter *MNPSM* is less than *circ. CA* (283), while *EFGKE* is greater than the *circ. OD*; therefore it is impossible that *CA* should be to *OB* as *circ. CA* is to a circumference less than *circ. OB*; or, in other words,

it is impossible that the radius of one circle should be to that of another as the circumference of the first is to a circumference less than that of the second.

It follows, moreover, from what has been said, that CA cannot be to OB as *circ.* CA is to a circumference greater than *circ.* OB; for, if this were the case, we should have by *inversion*,

$$OB : CA :: \text{a circumference greater than } circ.\ OB : circ.\ CA,$$

or, which is the same thing,

$$OB : CA :: circ.\ OB : \text{a circumference less than } circ.\ CA;$$

therefore the radius of one circle may be to the radius of another, as the circumference described upon the former is to a circumference less than the one described upon the latter, which has been shown to be impossible.

Since the fourth term of the proportion $CA : OB :: circ.\ CA : X$ can be neither less nor greater than *circ.* OB, it must be equal to *circ.* OB; therefore the circumferences of circles are as their radii.

By a construction and course of reasoning entirely similar, it may be demonstrated that the surfaces of circles are as the squares of their radii.

We shall not enter into further details upon this proposition, which is indeed a corollary from the next.

288. *Corollary.* Similar arcs AB, DE (*fig.* 166), are as their radii AC, DO; and similar sectors ACB, DOE, are as the squares of their radii. Fig. 166

For, since the arcs are similar, the angle C is equal to the angle O (163); now the angle C is to four right angles as the arc AB is to the entire circumference described upon the radius AC (122), and the angle O is to four right angles as the arc DE is to the circumference described upon the radius OD; therefore the arcs AB, DE, are to each other as the circumferences of which they are respectively a part; and these circumferences are as the radii AC, DO; therefore

$$arc.\ AB : arc\ DE :: AC : DO.$$

For the same reason the sectors ACB, DOE, are as the entire circles; but the entire circles are as the squares of their radii; therefore

$$sect.\ ACB : sect.\ DOE :: \overline{AC}^2 : \overline{DO}^2.$$

THEOREM.

289. *The area of a circle is equal to the product of its circumference by half of the radius.*

Demonstration. Denoting by *surf.* CA the surface or area of a circle whose radius is CA, we say that

$$surf.\ CA = \tfrac{1}{2}\ CA \times circ.\ CA.$$

Fig. 167. If $\frac{1}{2}\ CA \times circ.\ CA$ (*fig.* 167) be not the area of the circle of which CA is the radius, this quantity will be the measure of a circle either greater or less. Let us suppose, in the first place, that it is the measure of a greater circle, and that, if it be possible, $\frac{1}{2}\ CA \times circ.\ CA = surf.\ CB$.

About the circle, of which CA is the radius, circumscribe a regular polygon $DEFG$ &c., the sides of which shall not meet the circumference of the circle whose radius is CB (285); the surface of this polygon will be equal to its perimeter

$$DE + EF + FG + \&c.,$$

multiplied by $\frac{1}{2}\ AC$ (280). But the perimeter of the polygon is greater than that of the inscribed circle, since it encloses it on all sides; consequently the surface of the polygon $DEFG$ &c. is greater than $\frac{1}{2} AC \times circ.\ AC$, which, by hypothesis, is the measure of the circle, of which CB is the radius; hence the polygon would be greater than the circle; but it is less, since it is contained within it; therefore it is impossible that

$$\tfrac{1}{2} CA \times circ.\ CA$$

should be greater than *surf.* CA, or, in other words, it is impossible that the circumference of a circle multiplied by half of the radius should be the measure of a greater circle.

Again, this same product cannot be the measure of a less circle; and, not to change the figure, I will suppose that the circle in question is that whose radius is CB; it is to be proved then, that $\frac{1}{2}\ CB \times circ.\ CB$ cannot be the measure of a less circle, of the circle, for example, whose radius is CA. Let us suppose, if it be possible, that $\frac{1}{2} CB \times circ.\ CB = surf.\ CA$.

The same construction being supposed as above, the surface of the polygon $DEFG$ &c. will have for its measure

$$(DE + EF + FG + \&c.) \times \tfrac{1}{2}\ CA;$$

but the perimeter $DE + EF + FG + \&c.$, is less than *circ.* CB which encloses it on all sides; hence the area of the polygon is less than $\frac{1}{2} CA \times circ.\ CB$, and for a still stronger reason, less

than $\frac{1}{2}CB \times circ.\ CB$. This last quantity is, by hypothesis, the measure of the circle of which CA is the radius; consequently the polygon would be less than the inscribed circle, which is absurd; it is impossible then that the circumference of a circle multiplied by half of the radius should be the measure of a less circle.

Therefore the circumference of a circle multiplied by half of the radius is the measure of this circle.

290. *Corollary* I. The surface of a sector is equal to the arc of this sector multiplied by half of the radius.

For the sector ACB (*fig.* 168) is to the entire circle, as the arc AMB is to the entire circumference ABD (125), or as $AMB \times \frac{1}{2}AC$ is to $ABD \times \frac{1}{2}AC$. But the entire circle is equal to $ABD \times \frac{1}{2}AC$; therefore the sector ACB has for its measure $AMB \times \frac{1}{2}AC$. Fig. 1(

291. *Corollary* II. Since the circumferences of circles are as their radii, or as their diameters, calling π the circumference of a circle whose diameter is one, we have this proportion; the diameter of a circle 1 is to its circumference π, as the diameter $2CA$ is to the circumference of a circle whose radius is CA,

or $$1 : \pi :: 2CA : circ.\ CA;$$

hence $$circ.\ CA = 2\pi \times CA.$$

Multiplying each member by $\frac{1}{2}CA$, we have

$$\tfrac{1}{2}CA \times circ.\ CA = \pi \times \overline{CA}^2,$$

or $$surf.\ CA = \pi \times \overline{CA}^2;$$

therefore, *the surface of a circle is equal to the product of the square of the radius by the constant number* π, *which represents the circumference of a circle whose diameter is* 1, *or the ratio of the circumference to the diameter.*

In like manner, the surface of a circle whose radius is OB, is equal to $\pi \times \overline{OB}^2$. But

$$\pi \times \overline{CA}^2 : \pi \times \overline{OB}^2 :: \overline{CA}^2 : \overline{OB}^2;$$

therefore, *the surfaces of circles are to each other as the squares of their radii*, which agrees with the preceding theorem.

292. *Scholium.* We have already said, that the problem of the quadrature of the circle, consists in finding a square equal in surface to a circle whose radius is known; now we have just shown that a circle is equivalent to a rectangle contained by the

circumference and half of the radius, and this rectangle is changed into a square by finding a mean proportional between its two dimensions (243). Thus the problem of the quadrature of the circle reduces itself to finding the circumference, when the radius is known; and, for this purpose, it is sufficient to know the ratio of the circumference to the radius or to the diameter.

Hitherto we have not been able to obtain this ratio except by approximation; but the process has been carried so far, that a knowledge of the exact ratio would have no real advantage over the approximate ratio. Indeed this question, which occupied much of the attention of geometers, when the methods of approximation were less known, is now ranked among those idle questions which engage the attention of such only as have scarcely attained to the first principles of geometry.

Archimedes proved that the ratio of the circumference to the diameter is comprehended between $3\frac{10}{70}$ and $3\frac{10}{71}$; thus $3\frac{1}{7}$ or $\frac{22}{7}$ is a value already approaching very near to the number, which we have represented by π; and this first approximation is much in use on account of its simplicity. *Metius* gave a much nearer value of this ratio in the expression $\frac{355}{113}$. Other calculators have found the value of π, developed to a certain number of decimals, to be 3,1415926535897932 &c., and some have had the patience to extend these decimals to the hundred and twenty seventh, and even to the hundred and fortieth. Such an approximation may evidently be taken as equivalent to the truth, and the roots of imperfect powers are not better known.

We shall explain, in the following problems, two elementary methods, the most simple, for obtaining these approximations.

PROBLEM.

292. *The surface of a regular inscribed polygon and that of a similar circumscribed polygon being given, to find the surfaces of regular inscribed and circumscribed polygons of double the number of sides.*

Fig. 169. *Solution.* Let AB (*fig.* 169) be the side of a given inscribed polygon, EF parallel to AB, that of a similar circumscribed polygon, C the centre of the circle; if we draw the chord AM, and the tangents AP, BQ, the chord AM will be the side of an inscribed polygon of double the number of sides, and PQ double

of PM will be that of a similar circumscribed polygon (277); and, as the different angles of the polygon equal to ACM will admit of the same construction, it is sufficient to consider the angle ACM only, and the triangles here contained will be to each other as the entire polygons. Let A be the surface of the inscribed polygon whose side is AB, B the surface of a similar circumscribed polygon, A' the surface of a polygon whose side is AM, B' the surface of a similar circumscribed polygon. A and B are known, and it is proposed to find A' and B'.

1. The triangles ACD, ACM, the common vertex of which is A, are to each other as their bases CD, CM; moreover, these triangles are as the polygons A and A', of which they are respectively a part; hence

$$A : A' :: CD : CM.$$

The triangles CAM, CME, the common vertex of which is M, are to each other as their bases CA, CE; these same triangles are also as the polygons A' and B, of which they are respectively a part; hence

$$A' : B :: CA : CE.$$

But, on account of the parallels AD, ME,

$$CD : CM :: CA : CE;$$

therefore $$A : A' :: A' : B;$$

that is, the polygon A', one of those which is sought, is a mean proportional between the two known polygons A and B; consequently $$A' = \sqrt{A \times B}.$$

2. On account of the common altitude CM, the triangle CPM is to the triangle CPE as PM is to PE; but, as the line CP bisects the angle MCE (201),

$$PM : PE :: CM : CE :: CD : CA \text{ or } CM :: A : A';$$

hence $$CPM : CPE :: A : A',$$

and $$CPM : CPM + CPE \text{ or } CME :: A : A + A';$$

also $$2CPM \text{ or } CMPA : CME :: 2A : A + A'.$$

But $CMPA$ and CME are to each other as the polygons B' and B, of which they are respectively a part; we have then

$$B' : B :: 2A : A + A'.$$

Now A' has already been determined; and this new proportion will give the determination of B', namely,

$$B' = \frac{2A \times B}{A + A'};$$

therefore, by means of the polygons A and B, it is easy to find the polygons A' and B', which have double the number of sides.

PROBLEM.

294. *To find the approximate ratio of the circumference of a circle to its diameter.*

Solution. Let the radius of the circle be $=1$, the side of the inscribed square will be $\sqrt{2}$ (270), that of the circumscribed square will be equal to the diameter 2; hence the surface of the inscribed square $=2$, and that of the circumscribed square $=4$. Now, if we make $A=2$, and $B=4$, we shall find, by the preceding problem, the inscribed octagon $A'=\sqrt{8}=2{,}8284271$, and the circumscribed octagon $B'=\frac{16}{2+\sqrt{8}}=3{,}3137085$. Knowing thus the inscribed and circumscribed octagons, we can find, by means of them, the polygons of double the number of sides; we now suppose $A=2{,}8284271$, $B=3{,}3137085$, and we shall have $A'=\sqrt{A\times B}=3{,}0614674$, and $B'=\frac{2A\times B}{A+A'}=3{,}1825979$. These polygons of 16 sides will serve to find those of 32 sides, and we may proceed in this manner, till there is no difference between the inscribed and circumscribed polygons, at least for the number of decimals to which the calculation is carried, which in this example is seven. Having arrived at this point, we conclude that the circle is equal to the last result, for the circle must always be comprehended between the inscribed and circumscribed polygons; therefore, if these do not differ from each other for a certain number of decimals, the circle will not differ from them for the same number.

See the calculation of these polygons continued till they give the same result for the seven first decimals.

Number of sides.	Inscribed polygon.	Circumscribed polygon.
4	2,0000000	4,0000000
8	2,8284271	3,3137085
16	3,0614674	3,1825979
32	3,1214451	3,1517249
64	3,1365485	3,1441148
128	3,1403311	3,1422236
256	3,1412772	3,1417504
512	3,1415138	3,1416321
1024	3,1415729	3,1416025
2048	3,1415877	3,1415951
4096	3,1415914	3,1415933
8192	3,1415923	3,1415928
16384	3,1415925	3,1415927
32768	3,1415926	3,1415926

Hence we conclude that the surface of the circle = 3,1415926.

There might be some doubt with respect to the last decimal, on account of the error arising from the parts neglected; but we have extended the calculation to one decimal more in order to be assured of the correctness of the above result to the last figure.

Since the surface of a circle is equal to the product of the semicircumference by the radius, the radius being 1, the semicircumference will be 3,1415926; or, the diameter being 1, the circumference will be 3,1415926; therefore the ratio of the circumference to the diameter, above denoted by π, is equal to 3,1415926.

LEMMA.

295. *The triangle* CAB (fig. 170) *is equivalent to the isosceles triangle* DCE, *which has the same angle* C, *and of which the side* CE *equal to* CD *is a mean proportional between* CA *and* CB. *Moreover, if the angle* CAB *is a right angle, the perpendicular* CF *let fall upon the base of the isosceles triangle will be a mean proportional between the side* CA *and the half sum of the sides* CA, CB. Fig. 170

Demonstration. 1. On account of the common angle C, the triangle ABC is to the isosceles triangle DCE as $AC \times CB$ is to $DC \times CE$ or $\overline{DC}^2$ (216); consequently these triangles are equivalent, when $\overline{DC}^2 = AC \times CB$, or when DC is a mean proportional between AC and CB.

2. As the perpendicular CGF bisects the angle ACB,

$$AG : GB :: AC : CB \quad (201),$$

whence, by *composition*,

$$AG : AG + GB \text{ or } AB :: AC : AC + CB;$$

but $\quad AG : AB ::$ triangle ACG : triangle ACB or $2CDF$;

moreover, if the angle A is a right angle, the right-angled triangles ACG, CDF, are similar; whence

$$ACG : CDF :: \overline{AC}^2 : \overline{CF}^2;$$

or

$$ACG : 2CDF :: \overline{AC}^2 : 2\overline{CF}^2;$$

therefore

$$\overline{AC}^2 : 2\overline{CF}^2 :: AC : AC + CB.$$

If we multiply the two terms of the second ratio by AC, the antecedents will become equal, and we shall consequently have

$$2\overline{CF}^2 = AC \times (AC + CB), \text{ or } \overline{CF}^2 = AC \times \left(\frac{AC + CB}{2}\right);$$

therefore, if the angle A is a right angle, the perpendicular CF is a mean proportional between the side AC and half the sum of the sides AC, CB.

PROBLEM.

296. *To find a circle which shall differ as little as we please from a given regular polygon.*

Solution. Let there be given, for example, the square $BMNP$

Fig. 171. (*fig.* 171); let fall from the centre C the perpendicular CA upon the side MB, and join CB.

The circle described upon the radius CA is inscribed in the square, and the circle described upon the radius CB is circumscribed about this square; the first will be less than the square, and the second will be greater; it is proposed to reduce these limits.

Take CD and CE each equal to a mean proportional between CA and CB, and join ED; the isosceles triangle CDE will be equivalent to the triangle CAB (295); let the same be done with respect to each of the eight triangles which compose the square, and there will be formed a regular octagon equivalent to the square $BMNP$. The circle described upon CF, a mean proportional between CA and $\frac{CA + CB}{2}$, will be inscribed in the octagon, and the circle described upon CD, as a radius, will be circumscribed about it. Thus the first will be less and the second greater than the given square.

If we change, in the same manner, the right-angled triangle CDF into an equivalent isosceles triangle, we shall form in this way a regular polygon of sixteen sides equivalent to the proposed square. The circle inscribed in this polygon will be less than the square, and the circle circumscribed about it will be greater.

We can proceed in this manner till the ratio between the radius of the inscribed circle and that of the circumscribed circle shall differ as little as we please from equality. Then either of these circles may be regarded as equivalent to the proposed square.

297. *Scholium.* To exhibit the result of this investigation of the successive radii, let a be the radius of the circle inscribed in one of the polygons, and b the radius of the circle circumscribed about the same polygon; and let a', b', be similar radii to the next polygon

of double the number of sides. According to what has been demonstrated, b' is a mean proportional between a and b, and a' is a mean proportional between a and $\frac{a+b}{2}$; so that we have

$$b' = \sqrt{a \times b}, \text{ and } a' = \sqrt{a \times \frac{a+b}{2}};$$

hence the radii a and b of one polygon being known, the radii a', b', of the following polygon are easily deduced; and we may proceed in this manner till the difference between the two radii shall become insensible; then either of these radii may be taken for the radius of a circle equivalent to the proposed square or polygon.

This method may be readily applied to lines, since it consists in finding successive mean proportionals between known lines; but it succeeds still better by means of numbers, and it is one of the most convenient, that elementary geometry can furnish, for finding expeditiously the approximate ratio of the circumference of a circle to its diameter. Let the side of the square be equal to 2, the first inscribed radius CA will be 1, and the first circumscribed radius CB will be $\sqrt{2}$ or 1,4142136. Putting then $a = 1$, and $b = 1{,}4142136$, we shall have

$$b' = \sqrt{a \times b} = \sqrt{1 \times 1{,}4142136} = 1{,}1892071;$$

$$a' = \sqrt{a \times \frac{a+b}{2}} = \sqrt{1 \times \frac{1 + 1{,}4142136}{2}} = 1{,}0986841.$$

These numbers may be used in calculating the succeeding ones according to the law of continuation.

See the result of this calculation extended to seven or eight figures by means of a table of common logarithms.

Radii of the circumscribed circles.	Radii of the inscribed circles.
1,4142136	1,0000000.
1,1892071	1,0986841.
1,1430500	1,1210863.
1,1320149	1,1265639.
1,1292862	1,1279257.
1,1286063	1,1282657.

The first half of the figures being now the same in both, we can take the arithmetical instead of the geometrical means, since they do not differ from each other except in the remoter decimals (*Alg.* 102). The operation is thus greatly abridged, and the results are,

1,1284360	1,1283508.
1,1283934	1,1283721.
1,1283827	1,1283774.
1,1283801	1,1283787.
1,1283794	1,1283791.
1,1283792	1,1283792.

Hence 1,1283792 is very nearly the radius of a circle equal in surface to a square whose side is 2. From this it is easy to find the ratio of the circumference of a circle to its diameter; for it has been demonstrated that the surface of a circle is equal to the square of the radius multiplied by the number π; therefore if we divide the surface 4 by the square of 1,1283792, we shall have the value of π equal to 3,1415926 &c., as determined by the other method.

Appendix to the fourth section.

DEFINITIONS.

298. Among quantities of the same kind that which is greatest is called a *maximum*; and that which is smallest a *minimum*.

Thus the diameter of a circle is a *maximum* among all the straight lines drawn from one point of the circumference to another, and a perpendicular is a *minimum* among all the straight lines drawn from a given point to a given straight line.

299. Those figures which have equal perimeters are called *isoperimetrical figures*.

THEOREM.

300. *Among triangles of the same base and the same perimeter that is a* maximum *in which the two undetermined sides are equal.*

Fig. 172. *Demonstration.* Let $AC = CB$ (*fig.* 172), and

$$AM + MB = AC + CB;$$

the isosceles triangle ACB will be greater than the triangle AMB of the same base and the same perimeter.

From the point C, as a centre, and with the radius $CA = CB$, describe a circle meeting CA produced in D; join DB; and the angle DBA, inscribed in a semicircle is a right angle (128). Produce the perpendicular DB towards N, and make $MN = MB$, and join AN. From the points M and C let fall upon DN the

perpendiculars MP and CG. Since $CB = CD$, and $MN = MB$, $AC + CB = AD$, and $AM + MB = AM + MN$. But

$$AC + CB = AM + MB;$$

consequently $AD = AM + MN$; therefore $AD > AN$.

Now, if the oblique line AD is greater than the oblique line AN, it must be at a greater distance from the perpendicular AB (52); hence $DB > BN$, and BG the half DB is greater than BP the half BN. But the triangles ABC, ABM, which have the same base AB, are to each other as their altitudes BG, BP; therefore, since $BG > BP$, the isosceles triangle ABC is greater than the triangle ABM of the same base and same perimeter which is not isosceles.

THEOREM.

301. *Among polygons of the same perimeter and of the same number of sides, that is a* maximum *which has its sides equal.*

Demonstration. Let $ABCDEF$ (*fig.* 173) be the *maximum* Fig. 1 polygon; if the side BC is not equal to CD, make, upon the base BD, an isosceles triangle BOD, having the same perimeter as BCD, the triangle BOD will be greater than BCD (300), and consequently the polygon $ABODEF$ will be greater than $ABCDEF$; this last then will not be a *maximum* among all those of the same perimeter and the same number of sides, which is contrary to the supposition. Hence BC must be equal to CD; and, for the same reason, we shall have $CD = DE$, $DE = EF$, &c.; therefore all the sides of the *maximum* polygon are equal to each other.

THEOREM.

302. *Of all triangles formed with two given sides making any angle at pleasure with each other, the* maximum *is that in which the two given sides make a right angle.*

Demonstration. Let there be the two triangles BAC, BAD (*fig.* 174), which have the side AB common, and the side Fig. $AC = AD$; if the angle BAC is a right angle, the triangle BAC will be greater than the triangle BAD, in which the angle A is acute or obtuse.

For, the base AB being the same, the two triangles BAC, BAD, are as their altitudes AC, DE. But the perpendicular

DE is less than the oblique line *AD* or its equal *AC*; therefore the triangle *BAD* is less than *BAC*.

THEOREM.

303. *Of all polygons formed of given sides and one side to be taken of any magnitude at pleasure, the* maximum *must be such that all the angles may be inscribed in a semicircle of which the unknown side shall be the diameter.*

Fig. 175. *Demonstration.* Let *ABCDEF* (*fig.* 175), be the greatest of the polygons formed of the given sides *AB, BC, CD, DE, EF,* and the side *AF* taken at pleasure; draw the diagonals *AD, DF.* If the angle *ADF* is not a right angle, we can, by preserving the parts *ABCD, DEF,* as they are, augment the triangle *ADF*, and consequently the entire polygon by making the angle *ADF* a right angle, according to the preceding proposition; but this polygon can no longer be augmented, since it is supposed to have attained its *maximum*; therefore the angle *ADF* is already a right angle. The same may be said of the angles *ABF, ACF, AEF*; hence all the angles *A, B, C, D, E, F,* of the *maximum* polygon are inscribed in a semicircle of which the undetermined side *AF* is the diameter.

304. *Scholium.* This proposition gives rise to a question, namely, whether there are several ways of forming a polygon with given sides and one unknown side, the unknown side being the diameter of the semicircle in which the other sides are inscribed. Before deciding this question it is proper to observe that, if the same chord *AB* subtends arcs described upon differ-
Fig. 176. ent radii *AC, AD* (*fig.* 176), the angle at the centre subtended by this chord will be least in the circle of the greatest radius; thus $ACB < ADB$. For $ADO = ACD + CAD$ (78); therefore $ACD < ADO$, and, each being doubled, we have $ACB < ADB$.

THEOREM.

305. *There is but one way of forming a polygon* ABCDEF,
Fig. 175. (fig. 175) *with given sides and one side unknown, the unknown side being the diameter of the semicircle in which the others are inscribed.*

Demonstration. Let us suppose that we have found a circle which satisfies the question; if we take a greater circle, the chords *AB, BC, CD,* &c., answer to angles at the centre that are

smaller. The sum of the angles at the centre will accordingly be less than two right angles; thus the extremities of the given sides will not terminate in the extremities of a diameter. The contrary will occur if we take a smaller circle; therefore the polygon under consideration can be inscribed in only one circle.

306. *Scholium.* We can change at pleasure the order of the sides *AB*, *BC*, *CD*, &c., and the diameter of the circumscribed circle will always be the same as well as the surface of the polygon; for, whatever be the order of the arcs *AB*, *BC*, *CD*, &c., it is sufficient that their sum makes a semicircumference, and the polygon will always have the same surface, since it will be equal to the semicircle minus the segments *AB*, *BC*, *CD*, &c., the sum of which is always the same.

THEOREM.

307. *Of all polygons formed of given sides the* maximum *is that which can be inscribed in a circle.*

Demonstration. Let *ABCDEFG* (*fig.* 177) be an inscribed polygon, and *abcdefg* one that does not admit of being inscribed, having its corresponding sides equal to those of the former, namely, $ab = AB$, $bc = BC$, $cd = CD$, &c.; the inscribed polygon will be greater than the other. Fig. 177

Draw the diameter *EM*, and join *AM*, *MB*; upon $ab = AB$ construct the triangle *abm* equal to *ABM*, and join *em*.

According to what has just been demonstrated (303), the polygon *EFGAM* is greater than *efgam*, unless this last can also be inscribed in a semicircle having *em* for its diameter, in which case the two polygons would be equal (305). For the same reason the polygon *EDCBM* is greater than *edcbm*, with the exception of the case in which they are equal. Hence the entire polygon *EFGAMBCDE* is greater than *efgambcde*, unless they should be in all respects equal; but they are not so (161), since one is inscribed in a circle, and the other does not admit of being inscribed; therefore the inscribed polygon is greater than the other. Taking from them respectively the equal triangles *ABM*, *abm*, we have the inscribed polygon *ABCDEFG* greater than the polygon not inscribed *abcdefg*.

308. *Scholium.* It may be shown, as in art. 305, that there is only one circle and consequently only one *maximum* polygon

which satisfies the question; and this polygon will still have the same surface, whatever change be made in the order of the sides.

THEOREM.

309. *Among polygons of the same perimeter and the same number of sides the regular polygon is a* maximum.

Demonstration. According to art. 301, the *maximum* polygon has all its sides equal; and, according to the preceding theorem, it is such that it may be inscribed in a circle; therefore it is a regular polygon.

LEMMA.

310. *Two angles at the centre, measured in two different circles, are to each other as the contained arcs divided by their radii; that*
Fig. 178. *is, the angle* C : *angle* O : : *the ratio* $\frac{AB}{AC} : \frac{DE}{DO}$ (fig. 178).

Demonstration. With the radius OF equal to AC, describe the arc FG comprehended between the sides OD, OE, produced; on account of the equal radii AC, OF,

$$C : O :: AB : FG \quad (122), \text{ or } :: \frac{AB}{AC} : \frac{FG}{FO}.$$

But, on account of the similar arcs FG, DE,

$$FG : DE :: FO : DO \quad (288);$$

hence the ratio $\frac{FG}{FO}$ is equal to the ratio $\frac{DE}{DO}$; therefore

$$C : O :: \frac{AB}{AC} : \frac{DE}{DO}.$$

THEOREM.

311. *Of two regular isoperimetrical polygons that is the greater which has the greater number of sides.*

Fig. 179. *Demonstration.* Let DE (*fig.* 179), be half of a side of one of these polygons, O its centre, OE a perpendicular let fall from the centre upon one of the sides†; let AB be half of a side of the other polygon, C its centre, CB a perpendicular to the side let fall from the centre. We suppose the centres O and C to be

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† This perpendicular is called in the original *apothème.* No English word has been adopted answering to it.

situated at any distance OC, and the perpendiculars OE, CB, in the direction OC; thus DOE and ACB will be the semiangles at the centre of the polygons respectively, and as these angles are not equal, the lines CA, OD, being produced, will meet in some point F; from this point let fall upon OC the perpendicular FG; from the points O and C, as centres, describe the arcs GI, GH, terminating in the sides OF, CF.

This being done, we have, by the preceding lemma,

$$O : C :: \frac{GI}{OG} : \frac{GH}{CG};$$

but DE : perimeter of the first polygon :: O : four right angles, and AB : perimeter of the second polygon :: C : four right angles; hence, the perimeters of the polygon being equal,

$$DE : AB :: O : C$$

or

$$DE : AB :: \frac{GI}{OG} : \frac{GH}{CG}.$$

Multiplying the antecedents by OG and the consequents by CG, we have $\quad DE \times OG : AB \times CG :: GI : GH.$

But the similar triangles ODE, OFG, give

$$OE : OG :: DE : FG,$$

whence $\quad DE \times OG = OE \times FG;$

in like manner $\quad AB \times CG = CB \times FG;$

consequently $\quad OE \times FG : CB \times FG :: GI : GH,$

or $\quad OE : CB :: GI : GH.$

If therefore it is made evident that the arc GI is greater than the arc GH, it will follow that the perpendicular OE is greater than CB.

On the other side of CF let there be constructed the figure CKx equal to CGx, so that we may have $CK = CG$, the angle $HCK = HCG$, and the arc $Kx = xG$; the curve KxG enclosing the arc KHG will be greater than this arc (283). Hence Gx half of the curve is greater than GH half of the arc; therefore, for a still stronger reason, GI is greater than GH.

It follows from this that the perpendicular OE is greater than CB; but the two polygons having the same perimeter are to each other as these perpendiculars (280); therefore the polygon, which has for its half side DE, is greater than that which has for its half side AB. The first has the greater number of sides since its angle at the centre is less; therefore of two regular isoperimetrical polygons, that is the greater which has the greater number of sides.

THEOREM.

312. *The circle is greater than any polygon of the same perimeter.*

Demonstration. It has already been proved that among polygons of the same perimeter and the same number of sides, the regular polygon is the greatest; the inquiry is thus reduced to comparing the circle with regular polygons of the same perimeter. Let AI (*fig.* 180) be the half side of any regular polygon, and C its centre. Let there be, in the circle of the same perimeter, the angle $DOE = ACI$, and consequently the arc DE equal to the half side AI;

Fig. 180.

the polygon P : circle C : : triangle ACI : sector ODE,

hence $P : C : : \frac{1}{2}AI \times CI : \frac{1}{2}DE \times OE : : CI : OE.$

Let there be drawn to the point E the tangent EG meeting OD produced in G; the similar triangles ACI, GOE, give the proportion

$$CI : OE : : AI \text{ or } DE : GE;$$

therefore

$$P : C : : DE : GE : : DE \times \tfrac{1}{2}\, OE : GE \times \tfrac{1}{2}\, OE,$$

that is, $P : C$: : sector DOE : triangle GOE;

but the sector is less than the triangle; consequently P is less than C; therefore the circle is greater than any polygon of the same perimeter.

PART SECOND.

SECTION FIRST.

Of planes and solid angles.

DEFINITIONS.

313. A STRAIGHT line is *perpendicular to a plane*, when it is perpendicular to every straight line in the plane which passes through the foot of the perpendicular (326). Reciprocally, the plane, in this case, is perpendicular to the line.

The *foot* of the perpendicular is the point in which the perpendicular meets the plane.

314. A line is *parallel to a plane* when, each being produced ever so far, they do not meet. Also the plane, in this case, is parallel to the line.

315. Two *planes* are *parallel* when, being produced ever so far, they do not meet.

316. It will be demonstrated art. 324 that the common intersection of two planes, which meet each other, is a straight line. This being premised, the *angle* or the mutual *inclination of two planes* is the quantity, whether greater or less, by which they depart from each other; this quantity is measured by the angle contained by two straight lines drawn from the same point perperdicularly to the common intersection, the one being in one of the planes and the other in the other.

This angle may be acute, right, or obtuse.

317. If it is right, the two *planes* are *perpendicular* to each other.

318. A *solid angle* is the angular space comprehended between several planes which meet in the same point.

Thus the solid angle S (*fig.* 199) is formed by the meeting of Fig. 199 the planes ASB, BSC, CSD, DSA.

It requires at least three planes to form a solid angle.

14

THEOREM.

319. *One part of a straight line cannot be in a plane and another part without it.*

Demonstration. By the definition of a plane (6) a straight line, which has two points in common with the plane, lies wholly in that plane.

320. *Scholium.* In order to determine whether a surface is plane, it is necessary to apply a straight line in different directions to this surface and see of it touches the surface in its whole extent.

THEOREM.

321. *Two straight lines which cut each other are in the same plane, and determine its position.*

Fig. 181. *Demonstration.* Let AB, AC (*fig.* 181), be two straight lines which cut each other in A. Conceive a plane to pass through AB, and to be turned about AB, until it passes through the point C; then, two points A and C being in the plane, the whole line AC is in this plane; therefore the position of the plane is determined by the condition of its containing the two lines AB, AC.

322. *Corollary* I. A triangle ABC, or three points A, B, C, not in the same straight line determine the position of a plane.

Fig. 182. 323. *Corollary* II. Also two parallels AB, CD (*fig.* 182), determine the position of a plane; for, if the line EF be drawn, the plane of the two straight lines AE, EF, will be that of the parallels AB, CD.

THEOREM.

324. *If two planes cut each other, their common intersection is a straight line.*

Demonstration. If among the points common to the two planes there were three not in the same straight line, the two planes in question passing each through these three points would make only one and the same plane, which is contrary to the supposition.

THEOREM.

Fig. 183. 325. *If a straight line* AP (fig. 183) *is perpendicular to two others* PB, PC, *which intersect each other at its foot in the plane* MN, *it*

will be perpendicular to every other straight line PQ *drawn through its foot in the same plane, and thus it will be perpendicular to the plane* MN.

Demonstration. Through a point Q, taken at pleasure in PQ, draw the straight line BC in the angle BPC making $BQ = QC$ (242); join AB, AQ, AC.

The base BC being bisected at the point Q, the triangle BPC will give

$$\overline{PC}^2 + \overline{PB}^2 = 2\overline{PQ}^2 + 2\overline{QC}^2 \quad (194).$$

The triangle BAC will give, in like manner,

$$\overline{AC}^2 + \overline{AB}^2 = 2\overline{AQ}^2 + 2\overline{QC}^2.$$

If we subtract the first equation from the second, and recollect that the triangles APC, APB, each right-angled at P, give $\overline{AC}^2 - \overline{PC}^2 = \overline{AP}^2$, $\overline{AB}^2 - \overline{PB}^2 = \overline{AP}^2$; we shall have

$$\overline{AP}^2 + \overline{AP}^2 = 2\overline{AQ}^2 - 2\overline{PQ}^2;$$

or, by taking half of each member,

$$\overline{AP}^2 = \overline{AQ}^2 - \overline{PQ}^2;$$

hence

$$\overline{AP}^2 + \overline{PQ}^2 = \overline{AQ}^2;$$

therefore the triangle APQ is right-angled at P (193), and AP is perpendicular to PQ.

326. *Scholium.* It is evident then, not only that a straight line may be perpendicular to all those which pass through its foot in the plane, but that this happens, whenever the line in question is perpendicular to two straight lines drawn in the plane; hence the propriety of the definition art. 313.

327. *Corollary* I. The perpendicular AP is less than any oblique line AQ; therefore it measures the true distance of a point A from the plane PQ.

328. *Corollary* II. Through any given point P in a plane only one perpendicular can be drawn to this plane; for, if there could be two, a plane being supposed to pass through them intersecting the plane MN in PQ, the two perpendiculars would be perpendicular to the line PQ at the same point and in the same plane, which is impossible (50).

It is also impossible to let fall from a given point without a plane two perpendiculars to this plane; for let AP, AQ, be these two perpendiculars, then the triangle APQ would have two right angles APQ, AQP, which is impossible.

THEOREM.

329. ***Oblique lines equally distant from the perpendicular are equal; and of two oblique lines unequally distant from the perpendicular that which is at the greater distance is the greater.***

Fig 184. *Demonstration.* The angles APB, APC, APD (*fig.* 184), being right angles, if we suppose the distances PB, PC, PD, equal to each other, the triangles APB, APC, APD, have two sides and the included angle respectively equal, they are consequently equal; therefore the hypothenuses or the oblique lines AB, AC, AD, are equal to each other. Likewise if the distance PE is greater than PD or its equal PB, it is evident that the oblique line AE will be greater than AB or its equal AD.

330. *Corollary.* All the equal oblique lines AB, AC, AD, &c., terminate in the circumference of a circle BCD described about the foot of the perpendicular P, as a centre; therefore, a point A without a plane being given, to find the point P where the perpendicular A meets this plane, take three points B, C, D, equally distant from the point A, and find the centre of the circle which passes through these points, this centre will be the point P required.

331. *Scholium.* The angle ABP is called the *inclination of the oblique line* AB *to the plane* MN. It is manifest that this inclination is the same for all the oblique lines AB, AC, AD, &c., which depart equally from the perpendicular; for all the triangles ABP, ACP, ADP, &c., are equal.

THEOREM.

Fig. 185. 332. *Let* AP (fig. 185) *be a perpendicular to the plane* MN, *and* BC *a line situated in this plane; if from the foot* P *of the perpendicular a line* PD *be drawn perpendicular to* BC, *and* AD *be joined,* AD *will be perpendicular to* BC.

Demonstration. Take $DB = DC$, and join PB, PC, AB, AC. Since $DB = DC$, the oblique line $PB = PC$; and, because $PB = PC$, the oblique lines AB, AC, considered with reference to the perpendicular AP, are equal (329); hence the line AD has two points A and D each equally distant from the extremities B and C; therefore AD is perpendicular to BC (55).

333. *Corollary.* It will be seen, at the same time, that BC is perpendicular to the plane APD, since BC is perpendicular at the same time to the two straight lines AD and PD.

334 *Scholium.* The two lines AE, BC, present an example of two lines which do not meet, because they are not situated in the same plane. The least distance of these lines is the straight line PD, which is at the same time perpendicular to the line AP and to the line BC. The distance PD is the shortest; because, if we join two other points, as A and B, we shall have $AB > AD$, $AD > PD$, and, for a still stronger reason, $AB > PD$.

The two lines AE, CB, although not situated in the same plane, are considered as making a right angle with each other, because AD and a line drawn through any point in it parallel to BC, would make a right angle with each other. In like manner, the line AB and the line PD, which represent two straight lines not situated in the same plane, are considered as making the same angle with each other, as is made by AB and a line parallel to PD drawn through some point in AB.

THEOREM.

335. *If the line* AP (fig. 186) *is perpendicular to the plane* MN, *every line* DE *parallel to* AP *will be perpendicular to the same plane.* Fig. 186.

Demonstration. Let there be a plane passing through the parallels AP, DE, intersecting the plane MN in PD; in the plane MN draw BC perpendicular to PD, and join AD.

According to the corollary of the preceding theorem BC is perpendicular to the plane $APDE$; consequently the angle BDE is a right angle; but the angle EDP is also a right angle, since AP is perpendicular to PD, and DE is parallel to AP (65); hence the line DE is perpendicular to each of the lines DP, DB; therefore it is perpendicular to the plane MN passing through them (325).

336. *Corollary* I. Conversely, if the straight lines AP, DE, are perpendicular to the same plane MN, they will be parallel; for, if they are not, through the point D draw a line parallel to AP; this parallel will be perpendicular to the plane MN, consequently there would be two perpendiculars to the same plane drawn through the same point, which is impossible (328).

337. *Corollary* II. Two lines A and B, parallel to a third C, are parallel to each other; for, let there be a plane perpendicular to the line C, the lines A and B parallel to this perpendicu-

lar will be perpendicular to the same plane; therefore, by the above corollary, they are parallel to each other.

It is supposed that the three lines are not in the same plane, without which the proposition would already be known (68).

THEOREM.

Fig. 187. 338. *If the straight line* AB (fig. 187) *is parallel to a third line* CD, *drawn in the plane* MN, *it will be parallel to this plane.*

Demonstration. If the line *AB*, which is in the plane *ABCD*, should meet the plane *MN*, this can take place only in some point of the line *CD*, the common intersection of the two planes; now *AB* cannot meet *CD*. because it is parallel to it; consequently it cannot meet the plane *MN*; therefore it is parallel to this plane (314).

THEOREM.

Fig. 188. 339. *Two planes* MN, PQ (fig. 188), *perpendicular to the same straight line* AB, *are parallel to each other.*

Demonstration. If they can meet, let *O* be one of the common points of intersection, and join *OA*, *OB*; the line *AB*, perpendicular to the plane *MN*, is perpendicular to the straight line *OA* drawn through its foot in this plane; for the same reason, *AB* is perpendicular to *BO*; hence *OA*, *OB*, would be two perpendiculars let fall from the same point *O* upon the same straight line, which is impossible; consequently the planes *MN*, *PQ*, cannot meet; therefore they are parallel.

THEOREM.

Fig. 189. 340. *The intersections* EF, GH (fig. 189), *of two parallel planes* MN, PQ, *by a third plane* FG, *are parallel.*

Demonstration. If the lines *EF*, *GH*, situated in the same plane, are not parallel, being produced they will meet; consequently the planes *MN*, *PQ*, in which they are, would meet; therefore they would not be parallel.

THEOREM.

Fig. 188. 341. *The line* AB (fig. 188), *perpendicular to the plane* MN, *is perpendicular to the plane* PQ, *parallel to the plane* MN.

Demonstration. In the plane PQ draw at pleasure the line BC, and through AB, BC, suppose a plane ABC to pass intersecting the plane MN in AD, the intersection AD will be parallel to BC (340); but the line AB, perpendicular to the plane MN, is perpendicular to the straight line AD; consequently it will be perpendicular to its parallel BC; and, since the line AB is perpendicular to every line BC drawn through the foot of it in the plane PQ, it follows that it is perpendicular to the plane PQ.

THEOREM.

342. *The parallels* EG, FH (fig. 189), *comprehended between two parallel planes* MN, PQ, *are equal.* Fig. 189.

Demonstration. Through the parallels EG, FH, suppose a plane $EGHF$ to pass meeting the parallel planes in EF, GH. The intersections EF, GH, are parallel (340) as well as EG, FH; consequently the figure $EGHF$ is a parallelogram; therefore $EG = FH$.

343. *Corollary.* It follows from this, that *two parallel planes are throughout at the same distance from each other*; for, if EG, FH, are perpendicular to the two planes MN, PQ, they are parallel to each other (335); therefore they are equal.

THEOREM.

344. *If two angles* CAE, DBF (fig. 190), *not in the same plane, have their sides parallel and directed the same way, these angles will be equal, and their planes will be parallel.* Fig. 190.

Demonstration. Take $AC = BD$, $AE = BF$, and join CE, DF, AB, CD, EF. Since AC is equal and parallel to BD, the figure $ABDC$, is a parallelogram (87); therefore CD is equal and parallel to AB. For a similar reason, EF is equal and parallel to AB; consequently CD is also equal and parallel to EF; hence the figure $CEFD$ is a parallelogram, and thus the side CE is equal and parallel to DF; the triangles then DAE, DBF, are equilateral with respect to each other; therefore the angle

$$CAE = DBF.$$

Again, the plane ACE is parallel to the plane BDF; for, let us suppose the plane parallel to DBF, drawn through the point A, to meet the lines CD, EF, in points different from C and E, for example, in G and H; then, according to article 342, the

three lines AB, GD, FH, will be equal; but the three AB, CD, EF, are also equal; hence we should have $CD = GD$, and $FH = FE$, which is absurd; therefore the plane ACE is parallel to BDF.

345. *Corollary.* If two parallel planes MN, PQ, are met by two other planes $CABD$, $EABF$, the angles CAE, DBF, formed by the intersections in the parallel planes, are equal; for the intersection AC is parallel to BD (340), and AE to BF, therefore the angle $CAE = DBF$.

THEOREM.

346. *If three straight lines not in the same plane* AB, CD, EF
Fig. 190. (fig. 190), *are equal and parallel, the triangles* ACE, BDF, *formed by joining the extremities of these lines, on the one hand and on the other, will be equal and their planes will be parallel.*

Demonstration. Since AB is equal and parallel to CD, the figure $ABDC$ is a parallelogram; consequently the side AC is equal and parallel to BD. For a similar reason the sides AE, BF, are equal and parallel, as also CE, DF; hence the two triangles ACE, BDF, are equal; it may be shown moreover, as in the preceding proposition, that their planes are parallel.

THEOREM.

347. *Two straight lines comprehended between three parallel planes are divided into parts that are proportional to each other.*

Fig. 191. *Demonstration.* Let us suppose that the line AB (*fig.* 191) meets the parallel planes MN, PQ, RS, in A, E, B, and that the line CD meets the same planes in C, F, D, we shall have

$$AE : EB :: CF : FD.$$

Draw AD meeting the plane PQ in G, and join AC, EG, GF, BD; the intersections EG, BD, of the parallel planes PQ, RS, by the plane ABD, are parallel (340); hence, $AE : EB :: AG : GD$; and, because the intersections AC, GF, are parallel,

$$AG : GD :: CF : FD;$$

therefore, on account of the common ratio, $AG : GD$, we have

$$AE : EB :: CF : FD.$$

THEOREM.

Fig. 192. 348. *Let* ABCD (fig. 192) *be any quadrilateral either in the same plane or not; if the opposite sides are cut proportionally by*

two straight lines EF, GH, *so that* AE : EB : : DF : FC, and BG : GC : : AH : HD, *the straight lines* EF, GH, *will cut each other in a point* M, *in such a manner that* HM : MG : : AE : EB, EM : MF : : AH : HD.

Demonstration. Let there be any plane $AbHcD$ passing through AD which does not pass through GH; through the points E, B, C, F, draw Ee, Bb, Cc, Ff, parallel to GH meeting this plane in e, b, c, f. On account of the parallels Bb, GH, Cc,

$$bH : Hc :: BG : GC :: AH : HD; \quad (196);$$

consequently the triangles AHb, DHc, are similar (208). Also

$$Ae : eb :: AE : EB$$

and $$Df : fc :: DF : FC,$$

hence $$Ae : eb :: Df : fc,$$

or, by *composition* $$Ae : Df :: Ab : Dc;$$

but, on account of the similar triangles AHb, DHc,

$$Ab : Dc :: AH : HD,$$

consequently $$Ae : Df :: AH : HD.$$

Besides, the triangles AHb, DHc, being similar, the angle $HAe = HDf$; hence the triangles AHe, DHf, are similar (208), and consequently the angle $AHe = DHf$. It follows then, in the first place, that eHf is a straight line, and that thus the three parallels Ee, GH, Ff, are situated in the same plane which contains the two straight lines EF, GH; therefore *these must cut each other in a point* M. Moreover, on account of the parallels Ee, MH, Ff, $\quad EM : MF :: eH : Hf :: AH : HD.$

By a similar construction, referred to the side AB, it may be demonstrated that $HM : MG :: AE : EB$.

THEOREM.

346. *The angle comprehended between two planes* MAN, MAP, *may be measured, conformably to the definition, by the angle* PAN (fig. 193) *made by the two lines* AN, AP, *drawn one in one of these planes and the other in the other perpendicularly to the common intersection* AM. Fig. 193.

Demonstration. In order to show the legitimacy of this measure it is necessary to prove, 1. that it is constant, or in other words, that it is the same to whatever point of the common intersection the two perpendiculars are drawn.

If we take another point M, and draw MC in the plane MN, and MB in the plane MP, perpendicular to the common intersec-

tion AM; since MB and AP are perpendicular to the same line AM, they are parallel to each other. For the same reason MC is parallel to AN; consequently the angle $BMC = PAN$ (344); therefore, whether the perpendiculars be drawn to the point M or to the point A, the angle is always the same.

2. It is necessary to show that, if the angle of the two planes increases or diminishes, the angle PAN increases and diminishes in the same ratio.

In the plane PAN describe, from the centre A and with any radius, the arc NDP, and from the centre M and with the same radius, the arc CEB; draw AD to any point D in the arc NP; the two planes PAN, BMC, being perpendicular to the same straight line MA are parallel to each other (339); consequently the intersections AD, ME, of the two planes by the third AMD, are parallel; therefore the angle BME is equal to PAD (344).

Calling, for the present, the angle formed by the two planes MP, MN, a *wedge*, if the angle DAP were equal to DAN, it is evident that the wedge $DAMP$ would be equal to the wedge $DAMN$; for the base PAD might be applied exactly to its equal DAN, and the altitude AM would be the same for both; therefore the two wedges would coincide with each other. It is manifest, likewise, if the angle DAP were contained a certain number of times without a fraction in the angle PAN, the wedge $DAMP$ would be contained as many times in the wedge $PAMN$. Moreover, from a ratio in an entire number to any ratio whatever the conclusion is legitimate, and has been demonstrated in a case altogether similar (122); consequently, whatever be the ratio of the angle DAP to the angle PAN, the wedge $DAMP$ will have the same ratio to the wedge $PAMN$; therefore the angle NAP may be taken for the measure of the wedge $PAMN$, or of the angle made by the two planes MAP, MAN.

350. *Scholium.* It is with angles formed by two planes, as it is with angles formed by two straight lines. Thus, when two planes intersect each other, the angles opposite to each other at the vertex are equal, and the adjacent angles are together equal to two right angles; therefore, when one plane is perpendicular to another, the latter is perpendicular to the former. Also, when two parallel planes are intersected by a third plane, the same properties exist with respect to the angles thus formed, as take place, when two parallel lines are met by a third line (64).

THEOREM.

351. *The line* AP (fig. 194) *being perpendicular to the plane* MN, *any plane* APB, *passing through* AP, *will be perpendicular to the plane* MN. Fig. 194

Demonstration. Let *BC* be the intersection of the planes *AB*, *MN*; if, in the plane *MN*, the line *DE* be drawn perpendicular to *BP*, the line *AP*, being perpendicular to the plane *MN*, will be perpendicular to each of the two straight lines *BC*, *DE*. But the angle *APD* formed by the two perpendiculars *PA*, *PD*, at the common intersection *BP*, measures the angle of the two planes *AB*, *MN*; therefore, since this angle is a right angle, the two planes are perpendicular to each other (317).

352. *Scholium.* When three straight lines, as *AP*, *BP*, *DP*, are perpendicular to each other, each of these lines is perpendicular to the plane of the two others, and the three planes are perpendicular to each other.

THEOREM.

353. *If the plane* AB (fig. 194) *is perpendicular to the plane* MN, *and in the plane* AB *the line* AP *be drawn perpendicular to the common intersection* PB, *the line* AP *will be perpendicular to the plane* MN. Fig. 194

Demonstration. If, in the plane *MN*, the line *PD* be drawn perpendicular to *PB*, the angle *APD* will be a right angle, since the planes are perpendicular to each other; consequently, the line *AP* is perpendicular to the two straight lines *PB*, *PD*; therefore it is perpendicular to the plane *MN*.

354. *Corollary.* If the plane *AB* is perpendicular to the plane *MN*, and if through a point *P* of the common intersection a perpendicular to the plane MN be drawn, this perpendicular will be in the plane *AB*; for, if it is not, there may be drawn, in the plane *AB*, a line *AP* perpendicular to the common intersection *BP*, which would be at the same time perpendicular to the plane *MN*; therefore there would be two perpendiculars to the plane *MN* at the same point *P*, which is impossible (325).

THEOREM.

355. *If two planes* AB, AD (fig. 194), *are perpendicular to a third* MN, *their common intersection* AP *will be perpendicular to this third plane.* Fig. 194

Demonstration. If through the point P a perpendicular to the plane MN be drawn, this perpendicular must be at the same time in the plane AB and in the plane AD (354); therefore it is their common intersection AP.

THEOREM.

356. *If a solid angle is formed by three plane angles, the sum of either two of these angles will be greater than the third.*

Demonstration. We need consider only the case in which the plane angle to be compared with the two others is greater than
Fig. 195 either of them. Let there be then the solid angle S (*fig.* 195) formed by the three plane angles ASB, ASC, BSC, and let us suppose that the angle ASB is the greatest of the three; we say that $ASB < ASC + BSC$.

In the plane ASB make the angle $BSD = BSC$, draw at pleasure the straight line ADB; and, having taken $SC = SD$, join AC, BC.

The two sides BS, SD, are equal to the two BS, SC, and the angle $BSD = BSC$; hence the two triangles BSD, BSC are equal; consequently $BD = BC$. But $AB < AC + BC$; If we take from the one BD, and from the other its equal BC, there will remain $AD < AC$. The two sides AS, SD, are equal to the two AS, SC, and the third AD is less than the third AC; therefore the angle $ASD < ASC$ (42). Adding $BSD = BSC$ we shall have

$$ASD + BSD \text{ or } ASB < ASC + BSC.$$

THEOREM.

357. *The sum of the plane angles which form a solid angle is always less than four right angles.*

Fig. 196 *Demonstration.* Suppose the solid angle S (*fig.* 196) to be cut by a plane $ABCDE$; from a point O taken in this plane draw to the several angles the lines OA, OB, OC, OD, OE.

The sum of the angles of the triangles ASB, BSC, &c., formed about the vertex S, is equal to the sum of the angles of an equal number of triangles AOB, BOC, &c., formed about the vertex O. But, at the point B, the angles ABO, OBC, taken together, make the angle ABC less than the sum of the angles ABS, SBC (356); likewise, at the point C, $BCO + OCD < BCS + SCD$, and so on through all the angles of the polygon $ABCDE$. It follows then, that of the triangles whose vertex is in O the sum of the angles

at the bases is less than the sum of the angles at the bases of the triangles whose vertex is in S. Hence, the sum of the angles about the point O is greater than the sum of the angles about the point S. But the sum of the angles about the point O is equal to four right angles (34); therefore the sum of the plane angles, which form a solid angle S, is less than four right angles.

358. *Scholium.* It is supposed, in this demonstration, that the solid angle is convex, or that the plane of neither of the faces would, by being produced, cut the solid angle; if it were otherwise, the sum of the plane angles would no longer be limited, and might be of any magnitude whatever.

THEOREM.

359. *If two solid angles are respectively contained by three plane angles which are equal, each to each, the planes of any two of these angles in the one will have the same inclination to each other as the planes of the homologous angles in the other.*

Demonstration. Let the angle $ASC = DTF$ (*fig.* 197), the angle $ASB = DTE$, and the angle $BSC = ETF$; we say that the two planes ASC, ASB, will have, with respect to each other, an inclination equal to that of the planes DTF, DTE. Fig. 197.

Take SB of any magnitude, and draw BO perpendicular to the plane ASC; from the point O, where this perpendicular meets the plane, draw OA, OC, perpendicular respectively to SA, SC; join AB, BC. Take also $TE = SB$; and draw EP perpendicular to the plane DTF; from the point P draw PD, PF, perpendicular respectively to TD, TF; and join ED, EF.

The triangle SAB is right-angled at A, and the triangle TDE at D (332); and, since the angle $ASB = DTE$, we have also $SBA = TED$. Moreover, $SB = TE$; therefore the triangle $SAB = TDE$; consequently $SA = TD$, and $AB = DE$. It may be shown, in a similar manner, that $SC = TF$, and $BC = EF$. This being supposed, the quadrilateral $SAOC$ is equal to the quadrilateral $TDPF$; for, if we apply the angle ASC to its equal DTF, because $SA = TD$, and $SC = TF$, the point A will fall upon D, and the point C upon F. At the same time AO, perpendicular to SA, will fall upon DP, perpendicular to TD; and, in like manner, OC upon PF; therefore the point O will fall upon the point P, and we shall have $AO = DP$. But the triangles AOB, DPE, are right-angled at O and P, the hypothenuse $AB = DE$,

and the side $AO = DP$; consequently the triangles are equal (56); hence $OAB = PDE$. But the angle OAB is the inclination of the two planes ASB, ASC; and the angle PDE is the inclination of the two planes DTE, DTF; therefore these two inclinations are equal to each other.

It should be observed. however, that the angle A of the right-angled triangle OAB is not properly the inclination of the two planes ASB, ASC, except when the perpendicular BO falls, with respect to SA, on the same side as SC; if it should fall on the other side, the angle of the two planes would be obtuse, and, added to the angle A of the triangle OAB, it would make two right angles. But, in the same case, the angle of the two planes TDE, TDF, would be likewise obtuse, and, added to the angle D of the triangle DPE, it would make two right angles; therefore, as the angle A would be always equal to D, we infer, in like manner, that the inclination of the two planes ASB, ASC, is equal to that of the two planes TDE, TDF.

360. *Scholium.* If two solid angles are respectively contained by three plane angles which are equal, each to each, and if, at the same time, the angles of the one are *disposed in the same manner* as the homologous angles of the other, these solid angles will be equal, and, being applied the one to the other, will coincide. Indeed, we have already seen that the quadrilateral $SAOC$ may be placed upon its equal $TDPF$; thus, by placing SA upon TD, SC would fall upon TF, and the point O upon the point P. But, on account of the equality of the triangles AOB, DPE, the line OB perpendicular to the plane ASC is equal to PE perpendicular to the plane TDF; moreover the perpendiculars are directed the same way; therefore the point B will fall upon the point E, the line SB upon TE, and the two solid angles will coincide entirely with each other.

This coincidence, however, does not take place except by supposing that the plane angles *are disposed in the same manner* in each of the two solid angles; for if the plane angles were disposed *in a contrary order* in the one from what they are in the other; or, which comes to the same thing, if the perpendiculars OB, PE, instead of being directed the same way with respect to the planes ASC, DTF, were directed contrary ways, it would be impossible to make the solid angles coincide with

each other. Still it would not be the less true, that, agreeably to the theorem, the planes of the homologous angles would be equally inclined to each other; so that the two solid angles would be equal in all their constituent parts, without the property however of coinciding, when applied the one to the other. This kind of equality, which is not absolute, or does not admit of superposition, deserves to be distinguished by a particular denomination; we shall call it *equality by symmetry.*

Thus the two solid angles under consideration, which are respectively contained by three plane angles equal, each to each, but disposed in a contrary order in the one from what they are in the other, we shall call *angles equal by symmetry,* or simply *symmetrical angles.*

The same remark is applicable to solid angles contained by more than three plane angles; thus a solid angle contained by the plane angles A, B, C, D, E, and another solid angle contained by the same angles in the inverse order A, E, D, C, B, may be such that the planes of the homologous angles shall be equally inclined to each other. These two solid angles, which would be equal without admitting of superposition, we shall call *solid angles equal by symmetry,* or *symmetrical solid angles.*

There is not properly an equality by symmetry among plane figures; all those to which we might give this name, have the property of absolute equality, or equality by superposition. The reason is, that a plane figure may be reversed, and the upper side be taken for the under. It is not so with respect to solids, in which the third dimension may be taken in two different ways.

PROBLEM.

361. *Three plane angles forming a solid angle being given, to find, by a plane construction, the angle which two of these planes make with each other.*

Solution. Let S (*fig.* 198) be the proposed solid angle in which the three plane angles ASB, ASC, BSC, are known; the angle made by two of these planes with each other, ASB, ASC, for example, is required. Fig. 198.

The same construction being supposed as in the preceding theorem, the angle OAB would be the angle sought. It is proposed to find the same angle by a plane construction, or by lines traced upon a plane.

In order to this, make upon a plane the angles *B'SA*, *ASC*, *B''SC*, equal to the angles *BSA*. *ASC*, *BSC*, in the solid figure; take *B'S*, *B''S*, each equal to *BS* in the solid figure; from the points *B'* and *B''* let fall *B'A* and *B''C* perpendicularly upon *SA* and *SC*, which will meet in a point *O*. From the point *A*, as a centre, and with the radius *AB*, describe the semicircumference *B'bE*; at the point *O* erect upon *B'E* the perpendicular *Ob* meeting the circumference in *b*; join *Ab*, and the angle *EAb* will be the inclination sought of the two planes *ASC*, *ASB*, in the solid angle.

We have only to show that the triangle *AOb* of the plane figure is equal to the triangle *AOB* of the solid figure. Now the two triangles *B'SA*, *BSA*, are right-angled at *A*, and the angles at *S* are equal, consequently the angles at *B* and *B'* are also equal. But the hypothenuse *SB'* is equal to the hypothenuse *SB*; therefore the triangles are equal; hence *SA* in the plane figure is equal to *SA* in the solid figure, also *AB'* or its equal *Ab* in the plane figure, is equal to *AB* in the solid figure. It may be shown. in the same manner, that *SC* in one figure is equal to *SC* in the other; whence it follows that the quadrilateral *SAOC* in one figure is equal to *SAOC* in the other, and that thus *AO* in the plane figure is equal to *AO* in the solid figure; consequently the right-angled triangles *AOb*, *AOB*, have their hypothenuses equal and one side of the one equal to one side of the other; they are therefore equal, and the angle *EAb*, found by the plane construction, is equal to the inclination of the planes *SAB*, *SAC*, in the solid angle.

When the point *O* falls between *A* and *B'* in the plane figure, the angle *EAb* becomes obtuse, and always measures the true inclination of the planes. It is on this account that we have designated the required inclination by *EAb*, and not by *OAb*, in order that the same solution may be adapted to every case without exception.

362. *Scholium.* It may be asked, if, any three plane angles, taken at pleasure, can be made to form a solid angle.

In the first place, it is necessary that the sum of the three given angles should be less than four right angles, otherwise the solid angle could not be formed (357); it is necessary, moreover, that. after having taken two of the angles at pleasure *B'SA*, *ASC*, the third *CSB''* should be such that the perpendicu-

lar $B''C$ to the side SC shall meet the diameter $B'E$ between its extremities B' and E. Thus the limits of the magnitude of the angle CSB'' are such as require the perpendicular $B''C$ to terminate at the points B' and E. From these points let fall upon CS the perpendiculars $B'I$, EK, meeting in I and K the circumference described upon the radius SB'', and the limits of the angle CSB'' will be CSI and CSK.

But, in the isosceles triangle $B'SI$, the line CS produced being perpendicular to the base $B'I$,

$$\text{the angle } CSI = CSB' = ASC + ASB'.$$

And, in the isosceles triangle ESK the line SC being perpendicular to EK, the angle $CSK = CSE$. Moreover, on account of the equal triangles ASE, ASB', the angle $ASE = ASB'$; therefore CSE or $CSK = ASC - ASB'$.

Hence we infer that the problem will be possible, while the third angle is less than the sum of the two others ASC, ASB', and greater than their difference, a condition which accords with the theorem art. 356; for, by this theorem, we must have $CSB'' < ASC + ASB'$, also $ASC < CSB'' + ASB'$, or

$$CSB'' > ASC - ASB'.$$

PROBLEM.

363. *Two of the three plane angles, which form a solid angle, being given together with the angle which their planes make with each other, to find the third plane angle.*

Solution. Let ASC, ASB' (*fig.* 198), be the two given plane angles, and let us suppose, for the present, that CSB'' is the third angle sought; then, by constructing the figure as in the preceding problem, the angle contained by the planes of the two first would be EAb. Now, as we determine the angle EAb by means of CSB'', the two others being given; so we can determine CSB'' by means of EAb, and thus solve the proposed problem. Fig. 198

Having taken SB' of any magnitude at pleasure, let fall upon SA the indefinite perpendicular $B'E$, make the angle EAb equal to the angle of the two given planes; from the point b, where the side Ab meets the circumference described with the centre A and the radius AB', let fall upon AE the perpendicular bO, and from the point O let fall upon SC the indefinite perpendicular OCB'',

which terminate in B'' making $SB'' = SB'$; the angle CSB'' will be the third plane required.

For, if a solid angle be formed of the three planes $B'SA$, ASC, CSB'', the inclination of the planes containing the given angles ASB,' ASC, will be equal to the given angle EAb.

364. *Scholium.* If a solid angle is *quadruple*, or formed by
Fig. 199. four plane angles ASB, BSC, CSD, DSA (*fig.* 199), we cannot, by knowing these angles, determine the mutual inclination of their planes; for with the same plane angles any number of solid angles may be formed. But, if a condition be added, if, for example, the inclination of the two planes ASB, BSC, be given, then the solid angle is entirely determinate, and the inclination of any two of the planes may be found. Suppose a triple solid angle formed by the plane angles ASB, BSC, ASC; the two first angles are given as well as the inclination of their planes; we can then, by the problem just resolved, determine the third angle ASC. Afterward, if we consider the triple solid angle formed by the plane angles ASC, ASD, DSC, these three angles are known; thus the solid angle is entirely determinate. But the quadruple solid angle is formed by the union of the two triple solid angles of which we have been speaking; therefore, since these partial angles are known and determinate, the whole angle will be known and determinate.

The angle of the two planes ASD, DSC, may be found immediately by means of the second partial solid angle. As to the angle of the two planes BSC, CSD, it is necessary in one of the partial solid angles to find the angle comprehended between the two planes ASC, DSC, and in the other the angle comprehended between the two planes ASC, BSC; the sum of these angles will be the angle comprehended between the two planes BSC, DSC.

It will be found, in the same manner, that, in order to determine a quintuple solid angle, it is necessary to know, beside the five plane angles which compose it, two of the mutual inclinations of their planes; in a sextuple solid angle it is necessary to know three of these inclinations, and so on.

SECTION SECOND.

Of Polyedrons.

DEFINITIONS.

365. Every solid terminated by planes or plane faces is called a *solid polyedron*, or simply a *polyedron*. These planes are themselves necessarily terminated by straight lines.

A solid of four faces is called a *tetraedron*, one of six a *hexaedron*, one of eight an *octaedron*, one of twelve a *dodecaedron*, one of twenty an *icosaedron*, &c.

The tetraedron is the most simple of polyedrons; for it requires at least three planes to form a solid angle, and these three planes would leave an opening, to close which a fourth plane is necessary.

366. The common intersection of two adjacent faces of a polyedron is called a *side* or *edge* of the polyedron.

367. A *regular polyedron* is one, all whose faces are equal regular polygons, and all whose solid angles are equal to each other. There are five polyedrons of this kind.

368. A *prism* is a solid comprehended under several parallelograms terminated by two equal and parallel polygons.

To construct this solid let *ABCDE* (*fig.* 200) be any polygon; Fig. 200
if, in a plane parallel to *ABC* we draw the lines *FG*, *GH*, *HI*, &c., equal and parallel to the sides *AB*, *BC*, *CD*, &c., we shall form the polygon *FGHIK* equal to *ABCDE*; if now we connect the vertices of the homologous angles by the straight lines *AF*, *BG*, *CH*, &c., the faces *ABGF*, *BCHG*, &c., will be parallelograms, and the solid thus formed *ABCDEFGHIK* will be a prism.

369. The equal and parallel polygons *ABCD E FGHIK*, are called the *bases of the prism*. The other planes taken together, constitute the *lateral* or *convex surface of the prism*. The equal straight lines *AF*, *BG*, *CH*, &c., are called the *sides* of the prism.

370. The *altitude of a prism* is the distance between its bases, or the perpendicular let fall from a point in the superior base upon the plane of the inferior.

371. A *right prism* is one whose sides *AF*, *BG*, &c., are perpendicular to the planes of the bases; in this case, each of the

sides is equal to the altitude of the prism. In every other case the prism is *oblique*, and the altitude is less than the side.

372. A *prism* is *triangular, quadrangular, pentagonal, hexagonal*, &c., according as the base is a triangle, a quadrilateral, a pentagon, a hexagon, &c.

Fig. 206. 373. A prism whose base is a parallelogram (*fig.* 206), has all its faces parallograms, and is called a *parallelopiped.*

A *parallelopiped* is *rectangular*, when all its faces are rectangles.

374. Among rectangular parallelopipeds is distinguished the *cube* or regular hexaedron comprehended under six equal squares.

375. A *pyramid* is a solid formed by several triangular planes proceeding from the same point and terminating in the sides of a

Fig. 196. polygon $ABCDE$ (*fig.* 196).

The polygon $ABCDE$ is called the base of the pyramid, the point S its vertex, and the triangles ASB, BSC, &c., compose the *lateral* or *convex surface* of the pyramid.

376. The *altitude* of a pyramid is the perpendicular let fall from the vertex upon the plane of the base, produced if necessary.

377. A pyramid is *triangular, quadrangular*, &c., according as the base is a triangle, a quadrilateral, &c.

378. A pyramid is *regular*, when the base is a regular polygon, and the perpendicular, let fall from the vertex to the plane of the base, passes through the centre of this base. This line is called the *axis* of the pyramid.

379. The *diagonal* of a polyedron is a straight line which joins the vertices of two solid angles not adjacent.

380. I shall call *symmetrical polyedrons* two polyedrons which, having a common base, are similarly constructed, the one above the plane of this base and the other below it, with this condition, that the vertices of the homologous solid angles be situated at equal distances from the plane of the base, in the same straight line perpendicular to this plane.

Fig. 202. If the straight line ST (*fig.* 202), for example, is perpendicular to the plane ABC, and is bisected at the point O, where it meets this plane, the two pyramids $SABC$, $TABC$, which have the common base ABC, are two symmetrical polyedrons.

381. Two *triangular pyramids* are *similar* when they have two faces similar, each to each, similarly placed, and equally inclined to each other.

Thus, if we suppose the angle $ABC = DEF$, $BAC = EDF$, $ABS = DET$, $BAS = EDT$ (*fig.* 203), if also the inclination of the planes ABS, ABC, is equal to that of their homologous planes DTE, DEF, the pyramids $SABC$, $TDEF$, are similar. Fig. 203.

382. Having formed a triangle with the vertices of three angles, taken in the same face or base of a polyedron, we can imagine the vertices of the different solid angles of the polyedron, situated out of the plane of this base, to be the vertices of as many triangular pyramids, which have for their common base the above triangle; and these several pyramids will determine the positions of the several solid angles of the polyedron with respect to the base. This being supposed;

Two *polyedrons* are *similar*, when, the bases being similar, the vertices of the homologous solid angles are determined by triangular pyramids similar each to each.

383. I shall call *vertices* of a *polyedron* the points situated at the vertices of the different solid angles.

N. B. We shall consider only those polyedrons, which have saliant angles, or *convex* polyedrons. We thus denominate those, the surface of which cannot be met by a straight line in more than two points. In polyedrons of this description the plane of neither of the faces can, by being produced, cut the solid; it is impossible then, that the polyedron should be in part above the plane of one of the faces and in part below it; it is wholly on one side of this plane.

THEOREM.

384. *Two polyedrons cannot have the same vertices, the number also being the same, without coinciding the one with the other.*

Demonstration. Let us suppose one of the polyedrons already constructed, if we would construct another having the same vertices, the number also being the same, it is necessary that the planes of this last should not all pass through the same points as in the first; if they did, they would not differ the one from the other; but then it is evident that any new planes would cut the first polyedron; there would then be vertices above these planes and vertices below them, which does not consist with a convex polyedron; therefore, if two polyedrons have the same vertices, the number also being the same, they must necessarily coincide the one with the other.

Fig. 204. 385. *Scholium.* The points A, B, C, K, &c. (*fig.* 204), being given in position to be used as the vertices of a polyedron, it is easy to describe the polyedron.

Take, in the first place, three neighbouring points D, E, H, such that the plane DEH shall pass, if there is occasion for it, through other points K, C, but leaving all the rest on the same side, all above the plane, or all below it; the plane DEH or $DEHKC$, thus determined, will be a face of the solid. Through one of the sides EH of this face, suppose a plane to pass, and to turn upon this line until it meets a new vertex F, or several at the same time F, I; we shall thus have a second face FEH or $FEHI$. Proceed in this manner, by making planes to pass through the sides of the faces, until the solid is terminated in all directions; this solid will be the polyedron required, for there are not two which can have the same vertices.

THEOREM.

386. *In two symmetrical polyedrons the homologous faces are equal, each to each, and the inclination of two adjacent faces in one of the solids is equal to the inclination of the homologous faces in the other.*

Fig. 205. *Demonstration.* Let $ABCDE$ (*fig.* 205) be the common base of the two polyedrons, M and N the vertices of any two solid angles of one of the polyedrons, M' and N' the homologous vertices of the other polyedron; the straight lines MM', NN', must be perpendicular to the plane ABC, and be bisected at the points m and n (380), where they meet this plane. This being supposed, we say that the distance MN is equal to $M'N'$.

For, if the trapezoid $mM'N'n$ be made to revolve about mn, until its plane comes into the position of the plane $mMNn$, on account of the right angles at m and n, the side mM' will fall upon its equal mM, and nN' upon nN; therefore the two trapezoids will coincide, and we shall have $MN = M'N'$.

Let P be a third vertex in the superior polyedron, and P' the homologous vertex in the other, we shall have, in like manner, $MP = M'P'$, and $NP = N'P'$; consequently *the triangle* MNP, *formed by joining any three vertices of the superior polyedron is equal to the triangle* M'N'P', *formed by joining the homologous vertices of the other polyedron.*

If, among these triangles we consider only those which are formed at the surface of the polyedrons, we can conclude already that the surfaces of the two polyedrons are composed of the same number of triangles equal, each to each.

We say now that, if some of these triangles are in the same plane upon one surface and form the same polygonal face, the homologous triangles will be in the same plane upon the other surface and will form an equal polygonal face.

Let MPN, NPQ, be two adjacent triangles, which we suppose in the same plane, and let $M'P'N'$, $N'P'Q'$, be the homologous triangles. We have the angle $MNP = M'N'P'$, the angle $PNQ = P'N'Q'$; and, if we were to join MQ and $M'Q'$, the triangle MNQ would be equal to $M'N'Q'$, thus we should have the angle $MNQ = M'N'Q'$. But, since $MPNQ$ is one plane, we have the angle $MNQ = MNP + PNQ$; we have also

$$M'N'Q' = M'N'P' + P'N'Q'.$$

Now, if the three planes $M'N'P'$, $P'N'Q'$, $M'N'Q'$, are not confounded in one, they will form a solid angle, and we shall have the angle $M'N'Q' < M'N'P' + P'N'Q'$ (356); therefore, as this condition does not exist, the two triangles $M'N'P'$, $P'N'Q'$ are in the same plane.

We hence infer that each face, whether triangular, or polygonal, in one polyedron, corresponds to an equal face in the other, and that thus the two polyedrons are comprehended under the same number of planes equal, each to each.

It remains to show that the inclination of any two adjacent faces in one of the polyedrons is equal to the inclination of the two homologous faces in the other.

Let MPN, NPQ, be two triangles formed upon the common edge NP in the planes of two adjacent faces; let $M'P'N'$, $N'P'Q'$, be the homologous triangles. We can conceive at N a solid angle formed by the three plane angles MNQ, MNP, PNQ, and at N' a solid angle formed by the three $M'N'Q'$, $M'N'P'$, $P'N'Q'$. Now it has already been proved that these plane angles are equal, each to each; consequently the inclination of the two planes MNP, PNQ, is equal to that of their homologous planes $M'N'P'$, $P'N'Q'$ (359).

Therefore in symmetrical polyedrons the faces are equal, each to each, and the planes of any two adjacent faces of one of the solids have the same inclination to each other as the planes of the two homologous faces of the other solid.

387. *Scholium.* It may be remarked that *the solid angles of the one polyedron are symmetrical with the solid angles of the other;* for, if the solid angle N is formed by the planes MNP, PNQ, QNR, &c., its homologous angle N' is formed by the planes $M'N'P'$, $P'N'Q'$, $Q'N'R'$, &c. These last seem to be disposed in the same order as the others; but, as one of the solid angles is inverted with respect to the other, it follows that the actual disposition of the planes, which form the solid angle N', is the reverse of that which exists with respect to the homologous angle N. Moreover the inclinations of the successive planes in the one are equal respectively to those in the other; therefore these solid angles are symmetrical with respect to each other. See art. 360.

It will be perceived, from what has been said, that *any polyedron whatever can have only one polyedron symmetrical with it.* For, if there were constructed, upon another base, a new polyedron symmetrical with the given polyedron, the solid angles of this last would always be symmetrical with the angles of the given polyedron; consequently they would be equal to those of the symmetrical polyedron constructed upon the first base. Moreover, the homologous faces would always be equal; whence these two symmetrical polyedrons, constructed upon the one base and upon the other, would have their faces equal and their solid angles equal; therefore they would coincide by superposition, and would make one and the same polyedron.

THEOREM.

388. *Two prisms are equal, when three planes containing a solid angle of the one are equal to three planes containing a solid angle of the other, each to each, and are similarly placed.*

Fig. 200. *Demonstration.* Let the base $ABCDE$ (*fig.* 200), be equal to the base $abcde$, the parallelogram $ABGF$ equal to the parallelogram $abgf$, and the parallelogram $BCHG$ equal to the parallelogram $bchg$; we say that the prism $ABCI$ will be equal to the prism $abci$.

For, let the base $ABCDE$ be placed upon the base $abcde$, the two bases will coincide. But the three plane angles, which form the solid angle B, are equal to the three plane angles, which form the solid angle b, each to each, namely, $ABC = abc$, $ABG = abg$,

' $=$ *gbc*; also these angles are similarly placed; therefore olid angles *B* and *b* are equal (360), and consequently the *BG* will fall upon its equal *bg*. We see also that, on unt of the equal parallelograms *ABGF*, *abgf*, the side *GF* fall upon its equal *gf*, and likewise *GH* upon *gh*; therefore uperior base *FGHIK* will coincide entirely with its equal , and the two solids will form one and the same solid, since have the same vertices (384).

9. *Corollary. Two right prisms, which have equal bases equal altitudes, are equal.* For, since the side *AB* $=$ *ab*, and altitude *BG* $=$ *bg*, the rectangle *ABGF* $=$ *abgf*; the same be proved with respect to the rectangles *BGHC*, *bghc*; thus hree planes, which form the solid angle *B*, are equal to the which form the solid angle *b*, therefore the two prisms are l.

THEOREM.

0. *In every parallelopiped the opposite planes are equal and llel.*

emonstration. According to the definition of this solid, the s *ABCD*, *EFGH* (*fig.* 206), are equal parallelograms, and Fig. 206.
 sides are parallel (373). It remains then to demonstrate the same is true with respect to two opposite lateral faces, as *ID*, *BFGC*. Now *AD* is equal and parallel to *BC*, since the e *ABCD* is a parallelogram; for a similar reason *AE* is equal parallel to *BF*; consequently the angle *DAE* is equal to the e *CBF* (344), and the plane *DAE* parallel to *CBF*; therealso the parallelogram *DAEH* is equal to the parallelogram *'G*. In like manner it may be demonstrated that the oppoparallelograms *ABFE*, *DCGH*, are equal and parallel.

1. *Corollary.* Since a parallelopiped is a solid compreled under six planes of which the opposite ones are equal parallel, it follows that either of the faces and its opposite be taken for the bases of the parallelopiped.

2. *Scholium.* There being given three straight lines *AB*, *AD*, passing through the same point *A*, and making given es with each other; upon these three straight lines, a paropiped may be constructed; in order to this, a plane is to be e to pass through the extremity of each straight line parallel

to the plane of the two others; namely, through the point *B* a plane parallel to *DAE*, through the point *D* a plane parallel to *BAE*, and through the point *E* a plane parallel to *BAD*. The mutual meeting of these planes will form the parallelopiped required.

THEOREM.

393. *In every parallelopiped the opposite solid angles are symmetrical, and the diagonals drawn through the vertices of these angles bisect each other.*

Demonstration. Let us compare, for example, the solid angle
Fig. 206. *A* (*fig.* 206) with the solid angle *G*; the angle *EAB*, equal to *EFB*, is also equal to *HGC*, the angle $DAE = DHE = CGF$, and the angle $DAB = DCB = HGF$; consequently the three plane angles which form the solid angle *A*, are equal to the three, which form the solid angle *G*, each to each; besides, it is evident that their disposition in the one is different from that in the other; therefore the two solid angles *A* and *G* are symmetrical (359).

Again, let us suppose that the two diagonals *EC*, *AG*, to be drawn each through opposite vertices; since *AE* is equal and parallel to *CG*, the figure *AEGC* is a parallelogram; consequently the diagonals *EC*, *AG*, bisect each other. It may be demonstrated, in the same manner, that the diagonal *EC* and another *DF* also bisect each other; therefore the four diagonals bisect each other in a point which may be regarded as the centre of the parallelopiped.

THEOREM.

Fig. 207. 394. *The plane* BDHF (fig. 207), *which passes through two opposite parallel edges* BF, DH, *of a parallelopiped* AG *divides it into two triangular prisms* ABD-HEF, GHF-BCD, *symmetrical with each other.*

Demonstration. In the first place the solids are prisms; for the triangles *ABD*, *EFH*, having the sides of the one equal and parallel to those of the other, are equal; and, at the same time, the lateral faces *ABFE*, *ADHE*, *BDHF*, are parallelograms; therefore the solid *ABD-HEF* is a prism. The same may be proved with respect to the solid *GHF-BCD*. We say now that these two prisms are symmetrical with each other.

Upon the base *ABD* make the prism *ABD-E'F'H'* symmetrical with the prism *ABD-EFH*. According to what has been demonstrated (386), the plane *ABF'E'* is equal to *ABFE*, and the plane *ADH'E'* is equal to *ADHE*; but, if we compare the prism *GHF-BCD* with the prism *ABD-H'E'F'*, the base *GHF* is equal to *ABD*; the parallelogram *GHDC*, which is equal to *ABFE*, is also equal to *ABF'E'*, and the parallelogram *GFBC*, which is equal to *ADHE*, is also equal to *ADH'E'*; therefore the three planes, which form the solid angle *G* in the prism *GHF-BCD*, are equal to the three planes, which form the solid angle *A* in the prism *ABD-H'E'F'*, each to each; they are moreover similarly disposed in the two cases; therefore these two prisms are equal, and being applied the one to the other would coincide. But one of them is symmetrical with the prism *ABD-HEF*; therefore the other *GHF-BCD* is also symmetrical with *ABD-HEF*.

LEMMA.

395. *In every prism* ABCI *the sections* NOPQR, STVXY (fig. 201), *made by parallel planes are equal polygons.* Fig. 201.

Demonstration. The sides *NO*, *ST*, are parallel, being intersections of two parallel planes by a third plane *ABGF*; these same sides *NO*, *ST*, are comprehended between the parallels *NS*, *OT*, which are sides of the prism; consequently *NO* is equal to *ST*. For a similar reason, the sides *OP*, *PQ*, *QR*, &c., of the section *NOPQR* are equal respectively to the sides *TV*, *VX*, *XY*, &c., of the section *STVXY*. Besides, the equal sides being also parallel, it follows that the angles *NOP*, *OPQ*, &c., of the first section are equal respectively to the angles *STV*, *TVX*, &c., of the second (344). Therefore the two sections *NOPQR*, *STVXY*, are equal polygons.

396. *Corollary.* Every section made in a prism parallel to its base is equal to this base.

THEOREM.

397. *The two symmetrical triangular prisms* ABD-HEF, BCD-HFG (fig. 208), *which compose the parallelopiped* AG, *are equivalent.* Fig. 208.

Demonstration. Through the vertices *B*, *F*, perpendicular to the side *BF*, suppose the planes *Badc*, *Fehg* to pass, meeting the

three other sides AE, DH, CG, of the parallelopiped, the one in a, d, c, the other in e, h, g; the sections $Badc$, $Fehg$, will be equal parallelograms. They are equal, because they are made by planes, which are perpendicular to the same straight line, and consequently parallel (397); they are parallelograms, because the two opposite sides of the same section aB, dc, are the intersections of two parallel planes $ABFE$, $DCGH$, by the same plane.

For a similar reason, the figure $BaeF$ is a parallelogram, as also the other lateral faces $BFgc$, $cdhg$, $adhe$, of the solid $Badc\text{-}Fehg$; therefore this solid is a prism (368); and this prism is a right prism, since the side BF is perpendicular to the plane of the base.

This being premised, if the right prism Bh be divided by the plane $BFHD$ into two right triangular prisms $aBd\text{-}heF$, $Bdc\text{-}gFh$, we say that the oblique triangular prism $ABD\text{-}HEF$ will be equivalent to the right triangular prism $aBd\text{-}heF$.

Indeed, as the two prisms have the part $ABDheF$ common, it is necessary only to prove that the two remaining parts, namely, the solids $BaADd$, $FeEHh$, are equivalent to each other.

Now, on account of the parallelograms $ABFE$, $aBFe$, the sides AE, ae, being each equal to its parallel BF, are equal to each other; if then we take away the common part Ae, we shall have $Aa = Ee$. It may be shown, in like manner, that $Dd = Hh$.

Now, in order to apply the two solids $BaADd$, $FeEHh$, one to the other, let the base Feh be placed upon its equal Bad; the point e falling upon a, and the point h upon d, the sides eE, hH, will fall upon aA, dD, each upon its equal, since they are perpendicular to the same plane Bad; consequently the two solids under consideration will coincide entirely, the one with the other; therefore the oblique prism $BAD\text{-}HFE$ is equivalent to the right prism $Bad\text{-}hFe$.

It may be demonstrated, in like manner, that the oblique prism $BDC\text{-}GFH$ is equivalent to the right prism $Bdc\text{-}gFh$. But the two right prisms $Bad\text{-}Feh$, $Bdc\text{-}gFh$, are equal to each other, since they have the same alitude BF, and their bases Bad, Bdc, are each half of the same parallelogram (389). Therefore the two triangular prisms $BAD\text{-}HFE$, $BDC\text{-}GFH$, equivalent to equal prisms, are equivalent to each other.

398. *Corollary.* Every triangular prism *ABD-HFE* is half of the parallelopiped *AG*, constructed upon the same solid angle *A* with the same edges *AB*, *AD*, *AE*.

THEOREM.

399. *If two parallelopipeds* AG, AL (fig. 209), *have a common base* ABCD, *and have also their superior bases comprehended in the same plane and between the same parallels* EK, HL, *these two parallelopipeds will be equivalent.* Fig. 209.

Demonstration. The proposition admits of three cases, according as *EI* is greater than *EF*, less, or equal to it; but the demonstration is the same for each; and, in the first place, we say that the triangular prism *AEI-MDH* is equal to the triangular prism *BFK-LCG*.

Indeed, since *AE* is parallel to *BF*, and *HE* to *GF*, the angle *AEI* = *BFK*, *HEI* = *GFK*, *HEA* = *GFB*. Of these six angles the three first form the solid angle *E*, and the three last the solid angle *F*; consequently, since these plane angles are equal, each to each, and similarly disposed, it follows that the solid angles *E*, *F*, are equal. Now, if the prism *AEM* be applied to the prism *BFL*, the base *AEI* being placed upon the base *BFK*, these two bases, being equal, will coincide; and, since the solid angle *E* is equal to the solid angle *F*, the side *EH* will fall upon its equal *FG*. Nothing further is necessary in order to show that the two prisms will coincide throughout; for the base *AEI* and its edge *EH* determine the prism *AEM*, as the base *BFK* and its edge *FG* determine the prism *BFL* (388); therefore these prisms are equal.

But, if from the solid *AL* we take the prism *AEM*, there will remain the parallelopiped *AIL*; and, if from the same solid *AL* we take the prism *BFL*, there will remain the parallelopiped *AEG*; therefore the two parallelopipeds *AIL*, *AEG* are equivalent.

THEOREM.

400. *Two parallelopipeds which have the same base and the same altitude, are equivalent.*

Demonstration. Let *ABCD* (*fig.* 210) be the common base of two parallelopipeds *AG*, *AL*; since they have the same altitude, Fig. 210.

their superior bases *EFGH*, *IKLM*, will be in the same plane. Moreover the sides *EF*, *AB*, are equal and parallel, as also *IK*, *AB*; consequently *EF* is equal and parallel to *IK*; for a similar reason, *GF* is equal and parallel to *LK*. Produce the sides *EF*, *HG*, also *LK*, *MI*, till they shall, by their intersections, form the parallelogram *NOPQ*; it is evident that this parallelogram will be equal to each of the bases *EFGH*, *IKLM*. Now, if a third parallelopiped be supposed, which, with the same inferior base *ABCD*, has for its superior base *NOPQ*, this third parallelopiped will be equivalent to the parallelopiped *AG* (399); since, the inferior base being the same, the superior bases are comprehended in the same plane and between the same parallels *GQ*, *FN*. For the same reason, this third parallelopiped will be equivalent to the parallelopiped *AL*, therefore the two parallelopipeds *AG*, *AL*, which have the same base and the same altitude, are equivalent.

THEOREM.

401. *Every parallelopiped may be changed into an equivalent rectangular parallelopiped having the same altitude and an equivalent base.*

Fig. 210. *Demonstration.* Let *AG* (*fig.* 210) be the proposed parallelopiped; from the points *A*, *B*, *C*, *D*, draw *AI*, *BK*, *CL*, *DM*, perpendicular to the plane of the base, and we shall thus have the parallelopiped *AL* equivalent to the parallelopiped *AG*, and of which the lateral faces *AK*, *BL*, &c., will be rectangles. If then the base *ABCD* is a rectangle, *AL* will be the rectangular parallelopiped equivalent to the proposed parallelopiped *AG*.

Fig. 211. But, if *ABCD* (*fig.* 211) is not a rectangle, draw *AO*, *BN*, each perpendicular to *CD*, also *OQ*, *NP*, each perpendicular to the base, and we shall have the solid *ABNO-IKPQ*, which will be a rectangular parallelopiped. Indeed the base *ABNO* and the opposite base *IKPQ* are, by construction, rectangles; the lateral faces are also rectangles, since the edges *AI*, *OQ*, &c., are each perpendicular to the plane of the base; therefore the solid *AP* is a rectangular parallelopiped. But the two parallelopipeds *AP*, *AL*, may be considered as having the same base *ABKI*, and the same altitude *AO*; consequently they are equivalent; therefore the

Fig. 210, 211. parallelopiped *AG* (*fig.* 210, 211), which was first changed into an equivalent parallelopiped *AL*, is now changed into an equiva-

lent rectangular parallelopiped AP, which has the same altitude AI, and of which the base $ABNO$ is equivalent to the base $ABCD$.

THEOREM.

402. *Two rectangular parallelopipeds* AG, AL (fig. 212), *which have the same base* ABCD, *are to each other as their altitudes* AE, AI. Fig. 21

Demonstration. Let us suppose, in the first place, that the altitudes AE, AI, are to each other as two entire numbers, as 15 to 8, for example, AE may be divided into 15 equal parts of which AI will contain 8, and through the points of division x, y, z, &c., planes may be drawn parallel to the base. These planes will divide the solid AG into 15 partial parallelopipeds, which will be equal to each other, having equal bases and equal altitudes; we say equal bases, because every section of a prism $MIKL$, parallel to the base, is equal to this base (395), and equal altitudes, because the altitudes are the divisions themselves Ax, xy, yz, &c. Now, of these 15 equal parallelopipeds 8 are contained in AL; therefore the solid AG is to the solid AL as 15 is to 8, or in general as the altitude AE is to the altitude AI.

Again, if the ratio of AE to AI cannot be expressed in numbers, we say still, that the proportion is not the less true, namely,

$$\text{solid } AG : \text{solid } AL :: AE : AI,$$

For, if this proportion does not hold, let us suppose that

$$\text{solid } AG : \text{solid } AL :: AE : AO.$$

Divide AE into equal parts, each of which shall be less than IO; there will be at least one point of division m between I and O. Let P be the parallelopiped which has for its base $ABCD$ and for its altitude Am; since the altitudes AE, Am, are to each other as two entire numbers, we shall have

$$\text{solid } AG : P :: AE : Am.$$

But, by hypothesis,

$$\text{solid } AG : \text{solid } AL :: AE : AO,$$

whence $\quad \text{solid } AL : \quad P \quad :: AO : Am.$

But AO is greater than Am; it is necessary then, in order that this proportion may take place, that the solid AL should be greater than P; on the contrary it is less; consequently it is impossible that the fourth term of the proportion

$$\text{solid } AG : \text{solid } AL :: AE : x$$

should be a line greater than AI. By similar reasoning it may be shown that the fourth term cannot be less than AI; it is then equal to AI; therefore rectangular parallelopipeds of the same base are to each other as their altitudes.

THEOREM.

Fig. 213. 403. *Two rectangular parallelopipeds* AG, AK (fig. 213), *which have the same altitude* AE, *are to each other as their bases* ABCD, AMNO.

Demonstration. Having placed the two solids the one by the side of the other, as represented in the figure, produce the plane $ONKL$, till it meet the plane $DCGH$ in PQ, and a third parallelopiped AQ will be obtained, which may be compared with each of the parallelopipeds AG, AK. The two solids AG, AQ, having the same base $AEHD$ are to each other as their altitudes AB, AO; also the two solids AQ, AK, having the same base $AOLE$, are to each other as their altitudes AD, AM. Thus we have the two proportions

$$\text{solid } AG : \text{solid } AQ :: AB : AO,$$
$$\text{solid } AQ : \text{solid } AK :: AD : AM.$$

Multiplying the two proportions in order and omitting in the result the common multiplier *solid* AQ we shall have

$$\text{solid } AG : \text{solid } AK :: AB \times AD : AO \times AM.$$

But $AB \times AD$ represents the base $ABCD$ and $AO \times AM$ represents the base $AMNO$; therefore two rectangular parallelopipeds of the same altitude are to each other as their bases.

THEOREM.

404. *Any two rectangular parallelopipeds are to each other as the products of their bases by their altitudes, or as the products of their three dimensions.*

Fig. 213. *Demonstration.* Having placed the two solids AG, AZ (*fig.* 213), in such a manner that their surfaces may have a common angle BAE, produce the planes necessary to form the third parallelopiped AK of the same altitude with the parallelopiped AG, we shall have, by the preceding proposition,

$$\text{solid } AG : \text{solid } AK :: ABCD : AMNO.$$

But the two parallelopipeds AK, AZ, which have the same base $AMNO$, are to each other as their altitudes AE, AX; thus we

have $$\text{solid } AK : \text{solid } AZ :: AE : AX.$$
Multiplying these two proportions in order and omitting in the result the common multiplier *solid AK*, we obtain
$$\text{solid } AG : \text{solid } AZ :: ABCD \times AE : AMNO \times AX.$$
In the place of the bases $ABCD$, $AMNO$, we can substitute $AB \times AD$, $AO \times AM$, which will give
$$\text{solid } AG : \text{solid } AZ :: AB \times AD \times AE : AO \times AM \times AX.$$
Therefore any two rectangular parallelopipeds are to each other as the products of their bases by their altitudes, or as the products of their three dimensions.

405. *Scholium.* Hence we may take for the measure of a rectangular parallelopiped the product of its base by its altitude, or the product of its three dimensions. It is on this principle that we estimate all other solids.

In order to understand this measure it is necessary to recollect that by the product of two or several lines is meant the product of the numbers which represent these lines, and these numbers depend upon the linear unit, which may be taken at pleasure; the product therefore of the three dimensions of a parallelopiped is a number which of itself has no meaning, and which would be different according as one or another linear unit is used. But if, in like manner, the three dimensions of another parallelopiped are multiplied together, by estimating them according to the same linear unit, the two products would be to each other as the two parallelopipeds and would give an idea of their relative magnitude.

The magnitude of a solid, its volume, or its extension, constitutes what is called its *solidity*; and the word *solidity*† is employed particularly to denote the measure of a solid; thus we say that the solidity of a rectangular parallelopiped is equal to the product of its base by its altitude, or the product of its three dimensions.

The three dimensions of a cube being equal to each other, if the side is 1, the solidity will be $1 \times 1 \times 1$, or 1; if the side is

† *Content* is often employed by English writers to denote both solid and superficial measures. The word solidity, though most commonly used, is exceptionable, as it is likely to suggest to the mind of the student the idea of resistence. The term *volume* has been adopted by some as preferable to solidity.

2, the solidity will be $2 \times 2 \times 2$, or 8; if the side is 3, the solidity will be $3 \times 3 \times 3$, or 27, and so on; thus, the sides of cubes being as the numbers 1, 2, 3, &c., the cubes themselves, or their solidities, are as the numbers 1, 8, 27, &c. Hence the origin of what in arithmetic is called the *cube* of a number; it is the product arising from three factors, which are each equal to this number.

If it were proposed to make a cube double of a given cube, it would be necessary that the side of the cube sought should be to the side of the given cube as the cube root of 2 is to 1. Now it is easy to find, by a geometrical construction, the square root of 2; but we cannot, in this way, find the cube root of this number, at least by the simple operations of elementary geometry, which consist in employing only straight lines, two points of which are known, and circles whose centres and radii are determined.

On account of this difficulty the problem of the *duplication of the cube* was celebrated among the ancient geometers, as also that of the *trisection of an angle*, which is nearly of the same character. But the solutions, of which problems of this kind are susceptible, have long been known; and, although less simple than the constructions of elementary geometry, they are not less exact or less rigorous.

THEOREM.

406. *The solidity of a parallelopiped, and in general of any prism whatever, is equal to the product of its base by its altitude.*

Demonstration. 1. A parallelopiped of whatever kind is equivalent to a rectangular parallelopiped having the same altitude and an equivalent base (401). But the solidity of this last is equal to the product of its base by its altitude (405); therefore the solidity of the first is also equal to the product of its base by its altitude.

2. Every triangular prism is half of a parallelopiped, so constructed as to have the same altitude and a base twice as great (397). Now the solidity of this last is equal to the product of its base by its altitude (405); therefore the solidity of the triangular prism is equal to the product of its base, half of that of the parallelopiped, by its altitude.

3. A prism of whatever kind may be divided into as many triangular prisms of the same altitude, as there are triangles in the polygon taken for a base. But the solidity of each triangular prism is equal to the product of its base by its altitude; and, since the altitude is the same in each, it follows that the sum of all the partial prisms is equal to the sum of all the triangles, taken for bases, multiplied by the common altitude. Therefore the solidity of a prism of whatever kind is equal to the product of its base by its altitude.

407. *Corollary.* If we compare two prisms, which have the same altitude, the products of the bases by the altitudes will be as the bases; therefore *two prisms of the same altitude are to each other as their bases;* for a similar reason, *two prisms of the same base are to each other as their altitudes.*

LEMMA.

408. *If a pyramid* S-ABCDE (fig. 214) *is cut by a plane* abd, *parallel to the base,* Fig. 214

1. *The sides* SA, SB, SC, ... *and the altitude* SO, *will be divided proportionally in* a, b, c, ... *and* o;

2. *The section* abcde *will be a polygon similar to the base* ABCDE.

Demonstration. The planes ABC, abc, being parallel, their intersections AB, ab, by a third plane SAB, will be parallel (340); consequently the triangles SAB, Sab, are similar, and

$$SA : Sa :: SB : Sb;$$

in like manner $SB : Sb :: SC : Sc$,

and so on; therefore the sides SA, SB, SC, &c., are cut proportionally at a, b, c, &c. The altitude SO is cut in the same proportion at the point O; for BO and bo are parallel (340), and consequently

$$SO : So :: SB : Sb \quad (196).$$

2. Since ab is parallel to AB, bc to BC, cd to CD, &c., the angle $abc = ABC$, the angle $bcd = BCD$, and so on. Moreover, on account of the similar triangles SAB, Sab,

$$AB : ab :: SB : Sb;$$

and, on account of the similar triangles SBC, Sbc,

$$SB : Sb :: BC : bc;$$

whence $AB : ab :: BC : bc$;

in like manner, $BC : bc :: CD : cd$,
and so on. Therefore the polygons $ABCDE$, $abcde$, have their angles equal, each to each, and their homologous sides proportional; that is, they are similar.

409. *Corollary.* Let $S\text{-}ABCDE$, $S\text{-}XYZ$, be two pyramids that have a common vertex, and whose altitudes are the same, or whose bases are situated in the same plane; if these pyramids be cut by a plane parallel to their bases, *the sections* abcde, xyz, thus formed, *will be to each other as the bases* ABCDE, XYZ.

For, the polygons $ABCDE$, $abcde$, being similar, their surfaces are as the squares of their homologous sides AB, ab; but

$$AB : ab :: SA : Sa,$$

consequently $\quad ABCDE : abcde :: \overline{SA}^2 : \overline{Sa}^2.$

For the same reason,

$$XYZ : xyz :: \overline{SX}^2 : \overline{Sx}^2.$$

But, since $abcde$, xyz, are in the same plane,

$$SA : Sa :: SX : Sx,$$

whence $\quad ABCDE : abcde :: XYZ : xyz;$

therefore the sections $abcde$, xyz, are to each other as their bases $ABCDE$, XYZ.

LEMMA.

Fig. 215. 410. *Let* S-ABC (fig. 215), *be a triangular pyramid, of which* S *is the vertex and* ABC *the base; if the sides* SA, SB, SC, AB, AC, BC, *be bisected at the points* D, E, F, G, H, I, *and through these points the straight lines* DE, EF, DF, EG, FH, EI, GI, GH, *be drawn; we say that the pyramid* S-ABC *may be considered as composed of two prisms* AGH-FDE, EGI-CFH, *equivalent to each other, and two equal pyramids* S-DEF, E-GBI.

Demonstration. It follows from the construction, that ED is parallel to BA, and GE to AS (199); hence the figure $ADEG$ is a parallelogram. For the same reason, the figure $ADFH$ is also a parallelogram; consequently the straight lines AD, GE, HF, are equal and parallel; therefore the solid $AGH\text{-}FDE$ is a prism (346).

It may be shown, in like manner, that the two figures $EFCI$, $CIGH$, are parallelograms, and that thus the straight lines EF, IC, GH, are equal and parallel; therefore the solid $EGI\text{-}CFH$ is

a prism. Now we say that these two triangular prisms are valent to each other.

ıdeed, if upon the edges GI, GE, GH, the parallelopiped be formed, the triangular prism $EGI\text{-}CFH$ will be half his parallelopiped (397); on the other hand, the prism $H\text{-}FDE$ is also equal to half of the parallelopiped GX (406), e they have the same altitude, and the triangle AGH, the of the prism, is half of the parallelogram $GICH$ (168), base of the parallelopiped. Therefore the two prisms $I\text{-}CFH$, $AGH\text{-}FDE$, are equivalent to each other.

hese two prisms being taken from the pyramid $S\text{-}ABC$ there remain only the two pyramids $S\text{-}DEF$, $E\text{-}GBI$; now we say these two pyramids are equal to each other.

ıdeed, since the following sides are equal, namely, $BE = SE$, $= AG = DE$, $EG = AD = SD$, the triangle BEG is equal to triangle ESD (43). For a similar reason, the triangle BEI ıual to the triangle ESF; moreover the mutual inclination ıe two planes BEG, BEI, is the same as that of the two es ESD, ESF, since BEG, ESD, are in the same plane, BEI, ESF, are also in the same plane. If then, in order to y the one pyramid to the other, we place the triangle EBG ı its equal EDS, the plane BEI must fall upon the plane ESF; , since the triangles are equal and similarly disposed, the ıt I will fall upon F, and the two pyramids will coincide ıughout (384).

'herefore the entire pyramid $S\text{-}ABC$ is composed of two triular prisms AGF, GIF, equivalent to each other, and two ıl pyramids $S\text{-}DEF$, $E\text{-}GBI$.

11. *Corollary* I. From the vertex S let fall upon the plane C the perpendicular SO, and let P be the point, where this ıendicular meets the plane DEF, parallel to ABC; since $= \frac{1}{2}SA$, we have $SP = \frac{1}{2}SO$ (408), and the triangle $DEF = \frac{1}{4}$ ıgle ABC (218); consequently the solidity of the prism

$$AGH\text{-}FDE = \tfrac{1}{4}ABC \times \tfrac{1}{2}SO;$$

the solidity of the two prisms $AGH\text{-}FDE$, $EGI\text{-}CFH$, taken ther, is equal $\frac{1}{4}ABC \times SO$. These two prisms are less than pyramid $S\text{-}ABC$, since they are contained in it; therefore *olidity of a triangular pyramid is greater than the fourth part e product of its base by its altitude.*

412. *Corollary* II. If we join DG, DH, we shall have a new pyramid D-AGH equal to the pyramid S-DEF; for the base DEF may be placed upon its equal AGH, and then, the angles SDE, SDF, being equal to the angles DAG, DAH, it is manifest that DS will fall upon AD (364), and the vertex S upon the vertex D. Now the pyramid D-AGH is less than the prism AGH-FDE since it is contained in it; therefore each of the pyramids S-DEF, E-GBI, is less than the prism AGH-FDE; therefore the pyramid S-ABC, which is composed of two pyramids and two prisms, is less than four of these same prisms. But the solidity of one of these prisms $= \frac{1}{8}\ ABC \times SO$, and its quadruple $= \frac{1}{2}\ ABC \times SO$; hence *the solidity of any triangular pyramid is less than half of the product of its base by its altitude.*

THEOREM.

413. *The solidity of a triangular pyramid is equal to a third of the product of its base by its altitude.*

Fig. 215. *Demonstration.* Let S-ABC (*fig.* 215) be any triangular pyramid, ABC its base, SO its altitude; we say that the solidity of the pyramid S-ABC is equal to a third part of the product of the surface ABC by the altitude SO, so that

$$S\text{-}ABC = \tfrac{1}{3}\ ABC \times SO, \text{ or } = SO \times \tfrac{1}{3}\ ABC.$$

If this proposition be denied, the solidity S-ABC must be equal to the product of SO by a surface either greater or less than $\frac{1}{3}\ ABC$.

1. Let this quantity be greater, so that we shall have

$$S\text{-}ABC = SO \times (\tfrac{1}{3}\ ABC + M).$$

If we make the same construction as in the preceding proposition, the pyramid S-ABC will be divided into two equivalent prisms AGH-FDE, EGI-CFH, and two equal pyramids S-DEF, E-GBI. Now the solidity of the prism AGH-FDE is $DEF \times PO$, consequently we shall have the solidity of the two prisms

$$AGH\text{-}FDE + EGI\text{-}CFH = DEF \times 2PO, \text{ or } = DEF \times SO.$$

The two prisms being taken from the entire pyramid, the remainder will be equal to double of the pyramid S-DEF, so that we shall have

$$2S\text{-}DEF = SO \times (\tfrac{1}{3}\ ABC + M - DEF).$$

But, because SA is double of SD, the surface ABC is quadruple of DEF (408), and thus

$$\tfrac{1}{3} ABC - DEF = \tfrac{4}{3} DEF - DEF = \tfrac{1}{3} DEF;$$

ıce

$$2S\text{-}DEF = SO \times (\tfrac{1}{3} DEF + M),$$

y taking the half of each,

$$S\text{-}DEF = SP \times (\tfrac{1}{3} DEF + M).$$

pears then, that in order to obtain the solidity of the pyra- *S-DEF*, it is necessary to add to a third of the base the same ıce *M*, which was added to a third of the base of the large .mid, and to multiply the whole by the altitude *SP* of the l pyramid.

SD be bisected at the point *K*, and if through this point a plane *I* be supposed to pass parallel to *DEF* meeting the perpen- .ar *SP* in *Q*; according to what has just been demonstrated

$$S\text{-}KLM = SQ \times (\tfrac{1}{3} KLM + M).$$

re proceed thus to form a series of pyramids, the sides of h decrease in the ratio 2 to 1, and the bases in the ratio of 1, we shall soon arrive at a pyramid *S-abc*, the base of h *abc* shall be less than $6M$. Let *So* be the altitude of last pyramid; and its solidity, deduced from that of the eding pyramids, will be

$$So \times (\tfrac{1}{3} abc + M).$$

$M > \tfrac{1}{6} abc$, and consequently $\tfrac{1}{3} abc + M > \tfrac{1}{2} abc$. It would w then, that the solidity of the pyramid *S-abc* is greater than $\tfrac{1}{2} abc$; which is absurd, since it was proved in the preced- proposition, corollary II, that the solidity of a triangular ımid is always less than half of the product of its base by ıltitude; therefore it is impossible that the solidity of the mid *S-ABC* should be greater than $SO \times \tfrac{1}{2} ABC$.

Let *S-ABC* be equal to $SO \times (\tfrac{1}{3}ABC - M)$; it may be rn, as in the first case, that the solidity of the pyramid EF, the dimensions of which are less by one half, is equal to

$$SP \times (\tfrac{1}{3} DEF - M);$$

by continuing the series of pyramids, the sides of which ease in the ratio of 2 to 1, until we arrive at a term *S-abc* hall, in like manner, have the solidity of this last equal to

$$So \times (\tfrac{1}{3} abc - M).$$

as the bases *ABC*, *DEF*, *KLM* . . . . . *abc*, form a decreas- ıeries, each term of which is a fourth of the preceding, we soon arrive at a term *abc* equal to $12M$, or which shall ımprehended between $12M$ and $3M$; then, *M* being either

equal to or greater than $\frac{1}{12}$ *abc*, the quantity $\frac{1}{4}$ *abc* — *M* will either be equal to or less than $\frac{1}{6}$ *abc*; so that we shall have the solidity of the pyramid *S-abc* either equal to or less than

$$So \times \frac{1}{4}\,abc;$$

which is absurd, since, according to corollary I of the preceding proposition, the solidity of a triangular pyramid is always greater than the fourth of the product of its base by its altitude; therefore the solidity of the pyramid *S-ABC* cannot be less than $SO \times \frac{1}{3}\,ABC$.

We conclude then, according to the enunciation of the theorem, that the solidity of the pyramid $SABC = SO \times \frac{1}{3}\,ABC$, or $= \frac{1}{3}\,ABC \times SO$.

414. *Corollary* I. Every triangular pyramid is a third of a triangular prism of the same base and same altitude; for $ABC \times SO$ is the solidity of the prism of which *ABC* is the base and *SO* the altitude.

415. *Corollary* II. Two triangular pyramids of the same altitude are to each other as their bases, and two triangular pyramids of the same base are to each other as their altitudes.

THEOREM.

Fig. 214. 416. *Every pyramid* S-ABCDE (fig. 214) *has for its measure a third of the product of its base by its altitude.*

Demonstration. If the planes *SEB*, *SEC*, be made to pass through the diagonals *EB*, *EC*, the polygonal pyramid *S-ABCDE* will be divided into several triangular pyramids, which have all the same altitude *SO*. But, by the preceding theorem, these are measured by multiplying their bases *ABE*, *BCE*, *CDE*, each by a third of its altitude *SO*; consequently the sum of the triangular pyramids, or the polygonal pyramid *S-ABCDE* will have for its measure the sum of the triangles *ABE*, *BCE*, *CDE*, or the polygon *ABCDE*, multiplied by $\frac{1}{3}$ *SO*; therefore every pyramid has for its measure a third of the product of its base by its altitude.

417. *Corollary* I. Every pyramid is a third of a prism of the same base and same altitude.

418. *Corollary* II. Two pyramids of the same altitude are to each other as their bases, and two pyramids of the same base are to each other as their altitudes.

419. *Scholium.* The solidity of any polyedron may be estimated by decomposing it into pyramids, and this decomposition may be effected in several ways; one of the most simple is by means of planes of division passing through the vertex of the same solid angle; then we shall have as many partial pyramids, as there are faces in the polyedron excepting those which contain the solid angle from which the planes of division proceed.

THEOREM.

420. *Two symmetrical polyedrons are equivalent to each other or equal in solidity.*

Demonstration. 1. Two symmetrical triangular pyramids, as *S-ABC, T-ABC* (*fig.* 202), have each for its measure the product of the base *ABC* by a third of its altitude *SO* or *TO*; therefore these pyramids are equivalent. Fig. 202.

2. If we divide, in any manner, one of the symmetrical polyedrons in question into triangular pyramids, we can divide the other polyedron, in the same manner into triangular pyramids symmetrical with the former (382); but the triangular pyramids in the one case and the other being symmetrical, are equivalent, each to each; therefore the entire polyedrons are equivalent to each other or equal in solidity.

421. *Scholium.* This proposition seems to result immediately from a former (386), in which it was shown that, with respect to two symmetrical polyedrons, all the constituent parts of the one are equal respectively to those of the other; still it was necessary to demonstrate it in a rigorous manner.

THEOREM.

422. *If a pyramid is cut by a plane parallel to its base, the frustum which remains, after taking away the smaller pyramid, is equal to the sum of three pyramids, which have for their common altitude the altitude of the frustum, and whose bases are the inferior base of the frustum, its superior base, and a mean proportional between these bases.*

Demonstration. Let *S-ABCDE* (*fig.* 217) be a pyramid cut by the plane *abd* parallel to the base; let *T-FGH* be a triangular pyramid, whose base and altitude are equal or equivalent to the base and altitude of the pyramid *S-ABCDE*. The two bases Fig. 217

may be supposed to be situated in the same plane; and then the plane *abd* produced, will determine in the triangular pyramid a section *fgh* situated at the same altitude above the common plane of the bases; whence it follows that the section *fgh* is to the section *abd*, as the base *FGH* is to the base *ABD* (408); and, since the bases are equivalent, the sections will be equivalent also. Consequently the pyramids *S-abcde*, *T-fgh*, are equivalent, since they have the same altitude and equivalent bases. The entire pyramids *S-ABCDE*, *T-FGH*, are equivalent, for the same reason; therefore the frustums *ABD-dab*, *FGH-hfg*, are equivalent; and consequently it will be sufficient to demonstrate the proposition enunciated, with reference merely to the case of the frustum of a triangular pyramid.

Let *FGH-hfg* be the frustum of a triangular pyramid; through the points *F*, *g*, *H*, suppose a plane *FgH* to pass cutting off from the frustum the triangular pyramid *g-FGH*. This pyramid has for its base the inferior base *FGH* of the frustum, it has also for its altitude the altitude of the frustum, since the vertex *g* is in the plane of the superior base *fgh*.

This pyramid being cut off, there will remain the quadrangular pyramid *g-fhHF*, the vertex of which is *g* and the base *fhHF*. Through the three points *f*, *g*, *H*, suppose a plane *fgH* to pass dividing the quadrangular pyramid into two triangular pyramids *g-FfH*, *g-fhH*. This last pyramid may be considered as having for its base the superior base *gfh* of the frustum, and for its altitude the altitude of the frustum, since the vertex *H* is in the inferior base. Thus we have two of the three pyramids which compose the frustum.

It remains to consider the third pyramid *g-FfH*. Now if we draw *gK* parallel to *fF*, and suppose a new pyramid *K-FfH*, the vertex of which is *K*, and the base *FfH*; these two pyramids will have the same base *FfH*; they will have also the same altitude, since the vertices *g*, *K*, are situated in a line *gK* parallel to *Ff*, and consequently parallel to the plane of the base; therefore these pyramids are equivalent. But the pyramid *K-FfH* may be considered as having its vertex in *f*, and thus it will have the same altitude as the frustum; as to its base *FHK*, we say that it is a mean proportional between the two bases *FHG*, *fhg*. Indeed the triangles *FHK*, *fhg*, have the angle $F=f$, and the side $FK=fg$,

hence

$$FHK : fhg :: FK \times FH : fg \times fh :: FH : fh \quad (216).$$

Also $$FHG : FHK :: FG : FK \text{ or } fg.$$

But the similar triangles FHG, fhg, give

$$FG : fg :: FH : fh;$$

consequently $$FHG : FHK :: FHK : fhg;$$

and thus the base FHK is a mean proportional between the two bases FHG, fhg, therefore the frustum of a triangular pyramid is equal to three pyramids, which have for their common altitude the altitude of the frustum, and whose bases are the inferior base of the frustum, its superior base, and a mean proportional between these bases.

THEOREM.

423. *If a triangular prism, whose base is* ABC (fig. 216), *be cut by a plane* DEF *inclined to this base, the solid* ABC-DEF *thus formed, will be equal to the sum of the three pyramids whose vertices are* D, E, F, *and the common base* ABC. Fig. 21

Demonstration. Through the three points F, A, C, suppose a plane FAC to pass cutting off from the truncated prism

$$ABC\text{-}DEF$$

the triangular pyramid $F\text{-}ABC$; this pyramid will have for its base ABC and for its vertex the point F.

This pyramid being cut off, there will remain the quadrangular pyramid $F\text{-}ACDE$, of which F is the vertex, and $ACDE$ the base. Through the points F, E, C, suppose a plane FEC to pass dividing the quadrangular pyramid into two triangular pyramids $F\text{-}AEC$, $F\text{-}CDE$.

The pyramid $F\text{-}AEC$, which has for its base the triangle AEC and for its vertex the point F, is equivalent to a pyramid $B\text{-}AEC$, which has for its base AEC, and for its vertex the point B. For these two pyramids have the same base; they have also the same altitude, since the line BF, being parallel to each of the lines AE, CD, is parallel to their plane AEC; therefore the pyramid $F\text{-}AEC$ is equivalent to the pyramid $B\text{-}AEC$, which may be considered as having for its base ABC, and for its vertex the point E.

The third pyramid $F\text{-}CDE$, or $E\text{-}FCD$, may be changed in the first place into $A\text{-}FCD$; for the two pyramids have the same

base FCD; they have also the same altitude, since AE is parallel to the plane FCD; consequently the pyramid $E\text{-}FCD$ is equivalent to $A\text{-}FCD$. Again, the pyramid $A\text{-}FCD$, or $F\text{-}ACD$, may be changed into $B\text{-}ACD$, for these two pyramids have the common base ACD; they have also the same altitude, since their vertices F and B are in a parallel to the plane of the base. Therefore the pyramid $E\text{-}FCD$, equivalent to $A\text{-}FCD$, is also equivalent to $B\text{-}ACD$. Now this last may be regarded as having for its base ABC, and for its vertex the point D.

We conclude then, that the truncated prism $ABC\text{-}DEF$ is equal to the sum of three pyramids which have for their common base ABC and whose vertices are respectively the points D, E, F.

424. *Corollary.* If the edges are perpendicular to the plane of the base, they will be at the same time the altitudes of the three pyramids, which compose the truncated prism; so that the solidity of the truncated prism will be expressed by

$$\tfrac{1}{3} ABC \times AE + \tfrac{1}{3} ABC \times BF + \tfrac{1}{3} ABC \times CD,$$

or

$$\tfrac{1}{3} ABC \times (AE + BF + CD).$$

THEOREM.

425. *Two similar triangular pyramids have their homologous faces similar, and their homologous solid angles equal.*

Demonstration. The two triangular pyramids $S\text{-}ABC$, $T\text{-}DEF$

Fig. 203. (*fig.* 203), are similar, if the two triangles SAB, ABC, are similar to the two TDE, DEF, and are similarly placed (381); that is, if the angle $ABS = DET$, $BAS = EDT$, $ABC = DEF$, $BAC = EDF$, and if furthermore the inclination of the planes SAB, ABC, is equal to that of the planes TDE, DEF. This being supposed, we say that the pyramids have all their faces similar, each to each, and their homologous solid angles equal.

Take $BG = ED$, $BH = EF$, $BI = ET$, and join GH, GI, IH. The pyramid $T\text{-}DEF$ is equal to the pyramid $I\text{-}GBH$; for the sides GB, BH, being equal, by construction, to the sides DE, EF, and the angle GBH being, by hypothesis, equal to the angle DEF, the triangle GBH is equal DEF (36); therefore, in order to apply one of these pyramids to the other, we can evidently place the base DEF upon its equal GBH; then, since the plane TDE has the same inclination to DEF, as the plane SAB has to

ABC, it is manifest that the plane TDE will fall indefinitely upon the plane SAB. But, by hypothesis, the angle $DET = GBI$, consequently ET will fall upon its equal BI; and since the four points D, E, F, T, coincide with the four G, B, H, I, it follows that the pyramid $T\text{-}DEF$ will coincide with the pyramid $I\text{-}GBH$ (384).

Now, on account of the equal triangles DEF, GBH, the angle $BGH = EDF = BAC$; consequently GH is parallel to AC. For a similar reason GI is parallel to AS; therefore the plane IGH is parallel to SAC (344). Whence it follows that the triangle IGH, or its equal TDF, is similar to SAC (347), and that the triangle IBH, or its equal TEF, is similar to SBC; therefore the two similar triangular pyramids $S\text{-}ABC$, $T\text{-}DEF$ have their four faces similar, each to each. Moreover the homologous solid angles are equal.

For, we have already placed the solid angle E upon its homologous angle B, and the same may be done with respect to the two other homologous solid angles; but it will be readily perceived, that two homologous solid angles are equal, for example, the angles T and S, because they are formed by three plane angles which are equal, each to each, and similarly placed.

Therefore two similar triangular pyramids have their homologous faces similar and the homologous solid angles equal.

426. *Corollary* I. The similar triangles in the two pyramids furnish the proportions

$AB : DE :: BC : EF :: AC : DF :: AS : DT : SB : TE :: SC : TF$;

therefore *in similar triangular pyramids the homologous sides are proportional.*

427. *Corollary* II. Since the homologous solid angles are equal, it follows that *the inclination of any two faces of one pyramid is equal to the inclination of the two homologous faces of a similar pyramid* (359).

428. *Corollary* III. If a triangular pyramid $SABC$ be cut by a plane GIH parallel to one of the faces SAC, the partial pyramid $IGBH$ will be similar to the entire pyramid $SABC$. For the triangles BGI, BGH, are similar to the triangles BAS, BAC, each to each, and similarly placed; also the inclination of the two planes is the same in each; therefore the two pyramids are similar.

429. *Corollary* IV. *If any pyramid whatever* SABCDE (fig. 214) *be cut by a plane* abcde *parallel to the base, the partial* Fig. 214.

pyramid S-abcde *will be similar to the entire pyramid* S-ABCDE. For the bases $ABCDE$, $abcde$, are similar, and AC, ac, being joined, it has just been proved that the triangular pyramid S-ABC is similar to the pyramid S-abc; therefore the point S is determined with respect to the base ABC, as the point S is determined with respect to the base abc (382); therefore the two pyramids S-$ABCDE$, S-$abcde$, are similar.

430. *Scholium.* Instead of the five given things, required by the definition in order that two triangular pyramids may be similar, we can substitute five others, according to different combinations; and there will result as many theorems, among which may be distinguished the following; *two triangular pyramids are similar, when they have their homologous sides proportional.*

For, if we have the proportions

$AB : DE :: BC : EF :: AC : DF :: AS : DT :: SB : TE :: SC : TF$

Fig. 203. (*fig.* 203), which contain five conditions, the triangles ABS, ABC, will be similar to DET, DEF, and the disposition of the former will be similar to that of the latter. We have also the triangle SBC similar to TEF; therefore the three plane angles, which form the solid angle B, are equal to the three plane angles which form the solid angle E, each to each; whence it follows that the inclination of the planes SAB, ABC, is equal to that of the homologous planes TDE, DEF, and that thus the two pyramids are similar.

THEOREM.

431. *Two similar polyedrons have their homologous faces similar, and their homologous solid angles equal.*

Fig. 219. *Demonstration.* Let $ABCDE$ (*fig.* 219) be the base of one polyedron; let M, N, be the vertices of two solid angles, without this base, determined by the triangular pyramids M-ABC, N-ABC, whose common base is ABC; let there be, in the other polyedron, the base $abcde$ homologous or similar to $ABCDE$, m, n, the vertices homologous to M, N, determined by the pyramids m-abc, n-abc, similar to the pyramids M-ABC, N-ABC; we say, in the first place, that the distances MN, mn, are proportional to the homologous sides AB, ab.

Indeed, the pyramids M-ABC, m-abc, being similar, the inclination of the planes MAC, BAC, is equal to that of the planes

mac, *bac*; in like manner, the pyramids N-ABC, n-abc, being similar, the inclination of the planes NAC, BAC, is equal to that of the planes nac, bac; consequently, if we subtract the first inclinations respectively from the second, there will remain the inclination of the planes NAC, MAC, equal to that of the planes nac, mac. But, because the pyramids are similar, the triangle MAC is similar to mac, and the triangle NAC is similar to nac; therefore the triangular pyramids $MNAC$, $mnac$, have two faces similar each to each, similarly placed, and equally inclined to each other; consequently the two pyramids are similar (425); and their homologous sides give the proportion

$$MN : mn :: AM : am.$$

Moreover $$AM : am :: AB : ab;$$

therefore, $$MN : mn :: AB : ab.$$

Let P and p be two other homologous vertices of the same polyedrons, and we have, in like manner,

$$PN : pn :: AB : ab,$$
$$PM : pm :: AB : ab;$$

whence $$MN : mn :: PN : pn :: PM : pm.$$

Therefore *the triangle* PNM, *formed by joining any three vertices of one polyedron, is similar to the triangle* pnm, *formed by joining the three homologous vertices of the other polyedron.*

Furthermore, let Q, q, be two homologous vertices, and the triangle PQN will be similar to pqn. We say also, that the inclination of the planes PQN, PMN, is equal to that of the planes pqn, pmn.

For, if we join QM and qm, we shall always have the triangle QNM similar to qnm, and consequently the angle QNM equal to qnm. Suppose at N a solid angle formed by the three plane angles QNM, QNP, PNM, and at n a solid angle formed by the plane angles qnm, qnp, pnm; since these plane angles are equal, each to each, it follows that the solid angles are equal. Whence the inclination of the two planes PNQ, PNM, is equal to that of the homologous planes pnq, pnm (359); therefore, if the two triangles PNQ, PNM, be in the same plane, in which case we should have the angle $QNM = QNP + PNM$, we should have, in like manner, the angle $qnm = qnp + pnm$, and the two triangles qnp, pnm, would also be in the same plane.

All that has now been demonstrated takes place, whatever be

the angles M, N, P, Q, compared with the homologous angles m, n, p, q.

Let us suppose now that the surface of one of the polyedrons is divided into triangles ABC, ACD, MNP, NPQ, &c., we see that the surface of the other polyedron will contain an equal number of triangles abc, acd, mnp, npq, &c., similar to the former and similarly placed; and if several triangles, as MPN, NPQ, &c. belong to the same face, and are in the same plane, the homologous triangles mpn, npq, &c., will likewise be in the same plane. Therefore each polygonal face in the one polyedron will correspond to a similar polygonal face in the other; and consequently the two polyedrons will be comprehended under the same number of similar and similarly disposed planes. We say moreover, that the solid angles will be equal.

For, if the solid angle N, for example, is formed by the plane angles QNP, PNM, MNR, QNR, the homologous solid angle n will be formed by the plane angles qnp, pnm, mnr, qnr. Now the former plane angles are equal to the latter, each to each, and the inclination of any two adjacent planes, is equal to that of their homologous planes; therefore the two solid angles are equal, since they would coincide upon being applied.

We conclude then, that two similar polyedrons have their homologous faces similar, and their homologous solid angles equal.

432. *Corollary.* It follows, from the preceding demonstration, that if with four vertices of a polyedron we form a triangular pyramid, and also another with the four homologous vertices of a similar polyedron, these two pyramids will be similar; for they will have their homologous sides proportional (430).

It will be perceived, at the same time, that the homologous diagonals (157), AN, an, for example, are to each other as two homologous sides AB, ab.

THEOREM.

433. *Two similar polyedrons may be divided into the same number of triangular pyramids similar, each to each, and similarly placed.*

Demonstration. We have seen that the surfaces of two similar polyedrons may be divided into the same number of triangles,

that are similar, each to each, and similarly placed. Let us consider all the triangles of one of the polyedrons, except those which form the solid angle A, as the bases of so many triangular pyramids having their vertices in A; these pyramids taken together will compose the polyedron. Let us divide likewise the other polyedron into pyramids having for their common vertex that of the angle a, homologous to A; it is evident that the pyramid, which connects four vertices of one polyedron, will be similar to the pyramid which connects the four homologous vertices of the other polyedron; therefore two similar polyedrons, &c.

THEOREM.

434. *Two similar pyramids are to each other as the cubes of their homologous sides.*

Demonstration. Two pyramids being similar, the less may be placed in the greater so that they shall have the angle S (*fig.* 214) Fig. 214.
common. Then the bases $ABCDE$, $abcde$, will be parallel; for, since the homologous faces are similar (423), the angle

$$Sab = SAB,$$

as also $Sbc = SBC$; therefore the plane abc is parallel to the plane ABC (344). This being premised, let fall the perpendicular SO from the vertex S upon the plane ABC, and let o be the point, where this perpendicular meets the plane abc; we shall have, according to what has already been demonstrated (406),

$$SO : So :: SA : Sa :: AB : ab,$$

and consequently

$$\tfrac{1}{3} SO : \tfrac{1}{3} So :: AB : ab.$$

But the bases $ABCDE$, $abcde$, being similar figures,

$$ABCDE : abcde :: \overline{AB}^2 : \overline{ab}^2 \quad (221).$$

Multiplying the two proportions in order we shall have

$$ABCDE \times \tfrac{1}{3} SO : abcde \times \tfrac{1}{3} So :: \overline{AB}^3 :: \overline{ab}^3;$$

but $ABCDE \times \frac{1}{3} SO$ is the solidity of the pyramid $SABCDE$ (413), and $abcde \times \frac{1}{3} So$ is the solidity of the pyramid $Sabcde$; therefore two similar pyramids are to each other as the cubes of their homologous sides.

THEOREM.

435. *Two similar polyedrons are to each other as the cubes of their homologous sides.*

Demonstration. Two similar polyedrons may be divided into the same number of triangular pyramids that are similar, each to each (425). Now the two similar pyramids $APNM$, $apnm$
Fig. 219. (*fig.* 219), are to each other as the cubes of their homologous sides AM, am, or as the cubes of the homologous sides AB, ab (434. The same ratio may be shown to exist between any two other homologous pyramids; therefore the sum of all the pyramids, which compose the one polyedron, or the polyedron itself, is to the other polyedron, as the cube of any one of the sides of the first, is to the cube of the homologous side of the second.

General Scholium.

436. We can express in algebraic language, that is, in a manner the most concise, a recapitulation of the principal propositions of this section relating to the solidity or content of polyedrons.

Let B be the base of a prism, H its altitude; the solidity of the prism will be $B \times H$, or BH.

Let B be the base of a pyramid, H its altitude; the solidity of the pyramid will be $B \times \frac{1}{3} H$, or $H \times \frac{1}{3} B$, or $\frac{1}{3} BH$.

Let H be the altitude of the frustum of a pyramid and let A, B, be the bases; then $\sqrt{AB}$ will be the mean proportional between them, and the solidity of the frustum will be

$$\tfrac{1}{3} H \times (A + B + \sqrt{AB}).$$

Let B be the base of a truncated triangular prism, H, H', H'', the altitudes of the three superior vertices, the solidity of the truncated prism will be $\frac{1}{3} B \times (H + H' + H'')$.

Lastly, let P, p, be the solidities of two similar polyedrons, A and a two homologous sides, or diagonals, of the polyedrons, we shall have $\quad P : p :: A^3 : a^3.$

SECTION THIRD.

Of the Sphere.

DEFINITIONS.

437. A *sphere* is a solid terminated by a curved surface all the points of which are equally distant from a point within called the *centre.*

The sphere may be conceived to be generated by the revolution of a semicircle DAE (*fig.* 220) about its diameter DE; Fig. 220. for the surface thus described by the curve DAE will have all its points equally distant from the centre C.

438. The *radius of a sphere* is a straight line drawn from the centre to a point in the surface; the *diameter* or *axis* is a line passing through the centre and terminated each way by the surface.

All radii of the same sphere are equal; the diameters also are equal, and each double of the radius.

439. It will be demonstrated art. 452, that every section of a sphere made by a plane is a circle. This being supposed, we call a *great circle* the section made by a plane which passes through the centre, and a *small circle* the section made by a plane which does not pass through the centre.

440. A *plane* is a *tangent* to a sphere, when it has one point only in common with the surface of the sphere.

441. The *pole of a circle* of the sphere is a point in the surface of this sphere equally distant from every point in the circumference of the circle. It will be shown art. 464, that every circle great or small has two poles.

442. A *spherical triangle* is a part of the surface of a sphere comprehended by three arcs of great circles.

These arcs, which are called the *sides* of the triangle, are always supposed to be smaller each than a semicircumference. The angles which their planes make with each other are the angles of the triangle.

443. A spherical triangle takes the name of *right-angled, isosceles* and *equilateral,* like a plane triangle, and under the same circumstances.

444. A *spherical polygon* is a part of the surface of a sphere terminated by several arcs of great circles.

445. A *lunary surface* is the part of the surface of a sphere comprehended between two semicircumferences of great circles, which terminate in a common diameter.

446. We shall call a *spherical wedge* the part of a sphere comprehended between the halves of two great circles, and to which the lunary surface answers as a base.

447. A *spherical pyramid* is the part of sphere comprehended between the planes of a solid angle whose vertex is at the centre. The *base* of the pyramid is the spherical polygon intercepted by these planes.

448. A *zone* is the part of the surface of a sphere comprehended between two parallel planes which are its *bases*. One of these planes may be a tangent to the sphere, in which case the zone has only one base.

449. A *spherical segment* is the portion of a sphere comprehended between two parallel planes which are its bases. One of these planes may be a tangent to the sphere, in which case the spherical segment has only one base.

450. *The altitude of a zone* or *of a segment* is the distance between the parallel planes which are the bases of the zone or segment.

Fig. 220. 451. While the semicircle DAE (*fig.* 220), turning about the diameter DE, describes a sphere, every circular sector as DCF, or FCH, describes a solid which is called a *spherical sector*.

THEOREM.

452. *Every section of a sphere made by a plane is a circle.*

Fig. 221. *Demonstration.* Let AMB (*fig.* 221) be a section, made by a plane, of the sphere of which C is the centre. From the point C draw CO perpendicular to the plane AMB, and different oblique lines CM, CM, to different points of the curve AMB which terminates the section.

The oblique lines CM, CM, CB, are equal, since they are radii of the sphere; consequently they are at equal distances from the perpendicular CO (329); whence all the lines OM, OM, OB, are equal; therefore the section AMB is a circle of which the point O is the centre.

453. *Corollary* I. If the cutting plane pass through the centre of the sphere, the radius of the section will be the radius of the sphere; therefore all great circles are equal to each other.

454. *Corollary* II. Two great circles always bisect each other; for the common intersection, passing through the centre, is a diameter.

455. *Corollary* III. Every great circle bisects the sphere and its surface; for, if having separated the two hemispheres from each other we apply the base of the one to that of the other turning the convexities the same way, the two surfaces will coincide with each other; if they did not, there would be points in these surfaces unequally distant from the centre.

456. *Corollary* IV. The centre of a small circle and that of the sphere are in the same straight line perpendicular to the plane of the small circle.

457. *Corollary* V. Small circles are less according to their distance from the centre of the sphere; for, the greater the distance *CO*, the smaller the chord *AB*, the diameter of the small circle *AMB*.

458. *Corollary* VI. Through two given points on the surface of a sphere an arc of a great circle may be described; for the two given points and the centre of the sphere determine the position of a plane. If, however, the two given points be the extremities of a diameter, these two points and the centre would be in a straight line, and any number of great circles might be made to pass through the two given points.

THEOREM.

459. *In any spherical triangle* ABC (*fig.* 222) *either side is less than the sum of the other two.* Fig. 222.

Demonstration. Let *O* be the centre of the sphere, and let the radii *OA*, *OB*, *OC*, be drawn. If the planes *AOB*, *AOC*, *COB*, be supposed, these planes will form at the point *O* a solid angle, and the angles *AOB*, *AOC*, *COB*, will have for their measure the sides *AB*, *AC*, *BC*, of the sperical triangle *ABC* (123). But each of the three plane angles, which form the solid angle, is less than the sum of the two others (356); therefore either side of the triangle *ABC* is less than the sum of the other two.

THEOREM.

460. *The shortest way from one point to another on the surface of a sphere is the arc of a great circle which joins the two given points.*

Fig. 223. *Demonstration.* Let ANB (*fig.* 223) be the arc of a great circle which joins the two given points A and B, and let there be without this arc, if it be possible, a point M of the shortest line between A and B. Through the point M draw the arcs of great cirles MA, MB, and take $BN = MB$.

According to the preceding theorem the arc ANB is less than $AM + MB$; taking from one BN, and from the other its equal BM, we shall have $AN < AM$. Now the distance from B to M, whether it be the same as the arc BM, or any other line, is equal to the distance from B to N; for, by supposing the plane of the great circle BM to turn about the diameter passing through B, the point M may be reduced to the point N, and then the shortest line from M to B, whatever it may be, is the same as that from N to B; consequently the two ways from A to B, the one through M and the other through N, have the part from M to B equal to that from N to B. But the first way is, by hypothesis, the shortest; consequently the distance from A to M is less than the distance from A to N, which is absurd, since the arc AM is greater than AN; whence no point of the shortest line between A and B can be without the arc ANB; therefore this line is itself the shortest that can be drawn between its extremities.

THEOREM.

461. *The sum of the three sides of a spherical triangle is less than the circumference of a great circle.*

Fig. 224. *Demonstration.* Let ABC (*fig.* 224) be any spherical triangle; produce the sides AB, AC, till they meet again in D. The arcs ABD, ACD, will be the semicircumferences of great circles, since two great circles always bisect each other (454); but in the triangle BCD the side $BC < BD + CD$ (459); adding to each $AB + AC$ we shall have $AB + AC + BC < ABD + ACD$, that is, less than the circumference of a great circle.

THEOREM.

462. *The sum of the sides of any spherical polygon is less than the circumference of a great circle.*

Demonstration. Let there be, for example, the pentagon *ABCDE* (*fig.* 225); produce the sides *AB*, *DC*, till they meet in *F*; since *BC* is less than $BF + CF$, the perimeter of the pentagon *ABCDE* is less than that of the quadrilateral *AEDF*. Again produce the sides *AE*, *FD*, till they meet in *G*, and we shall have $ED < EG + GD$; consequently the perimeter of the quadrilateral *AEDF* is less than that of the triangle *AFG*; but this last is less than the circumference of a great circle (461); therefore for a still stronger reason the perimeter of the polygon *ABCDE* is less than this same circumference. Fig. 225.

463. *Scholium.* This proposition is essentially the same as that of art. 357; for if *O* be the centre of the sphere, we can suppose at the point *O* a solid angle formed by the plane angles *AOB*, *BOC*, *COD*, &c., and the sum of these angles must be less than four right angles, which does not differ from the proposition enunciated above; but the demonstration just given is different from that of art. 357; it is supposed in each that the polygon *ABCDE* is convex, or that no one of the sides produced would cut the figure.

THEOREM.

464. *If the diameter* DE (fig. 220) *be drawn perpendicular to the plane of the great circle* AMB, *the extremities* D *and* E *of this diameter will be the poles of the circle* AMB, *and of every small circle* FNG *which is parallel to it.* Fig. 220.

Demonstration. *DC*, being perpendicular to the plane *AMB*, is perpendicular to all the straight lines *CA*, *CM*, *CB*, &c., drawn through its foot in this plane (325); consequently all the arcs *DA*, *DM*, *DB*, &c., are quarters of a circumference. The same may be shown with respect to the arcs *EA*, *EM*, *EB*, &c., whence the points *D*, *E*, are each equally distant from all the points in the circumference of the circle *AMB*; therefore they are the poles of this circle (441).

Again, the radius *DC*, perpendicular to the plane *AMB* is perpendicular to its parallel *FNG*; consequently it passes

through the centre O of the circle FNG (456); whence, if DF, DN, DG, be drawn, these oblique lines will be equally distant from the perpendicular DO, and will be equal (329). But, the chords being equal, the arcs are equal; consequently all the arcs DF, DN, DG, &c., are equal; therefore the point D is the pole of the small circle FNG, and for the same reason the point E is the other pole.

465. *Corollary* I. Every arc DM, drawn from a point in the arc of a great circle AMB to its pole, is the fourth part of the circumference, which for the sake of conciseness we shall call a *quadrant*; and this quadrant at the same time makes a right angle with the arc AM. For the line DC being perpendicular to the plane AMC, every plane DMC, which passes through the line DC, is perpendicular to the plane AMC (351); therefore the angle of these planes, or, according to the definition art. 442, the angle AMD is a right angle.

466. *Corollary* II. In order to find the pole of a given arc AM, draw the indefinite arc MD perpendicular to AM, take MD equal to a quadrant, and the point D will be one of the poles of the arc MD; or rather, draw to the two points A, M, the arcs AD, MD, perpendicular each to AM, the point of meeting D of these two arcs will be the pole required.

467. *Corollary* III. Conversely, if the distance of the point D from each of the points A, M, is equal to a quadrant, we say that the point D will be the pole of the arc AM, and that, at the same time, the angles DAM, AMD, will be right angles.

For, let C be the centre of the sphere, and let the radii CA, CD, CM, be drawn. Since the angles ACD, MCD, are right angles, the line CD is perpendicular to the two straight lines CA, CM; whence it is perpendicular to their plane (325); therefore the point D is the pole of the arc AM; and consequently the angles DAM, AMD, are right angles.

468. *Scholium.* By means of poles, arcs may be traced upon the surface of a sphere as easily as upon a plane surface. We see, for example, that by turning the arc DF, or any other line of the same extent about the point D, the extremity F will describe the small circle FNG; and, by turning the quadrant DFA about the point D, the extremity A will describe the arc of a great circle AM.

If the arc AM is to be produced, or if only the points A, M, are given, through which this arc is to pass, we determine, in the first place, the pole D by the intersection of two arcs described from the points A, M, as centres with an extent equal to a quadrant. The pole D being found, we describe from the point D, as a centre, and with the same extent, the arc AM or the continuation of it.

If it is required to let fall a perpendicular from a given point P upon a given arc AM, we produce this arc to S, so that the distance PS shall be equal to a quadrant; then from the pole S and with the distance PS we describe the arc PM, which will be the perpendicular arc required.

THEOREM.

469. *Every plane perpendicular to the radius at its extremity is a tangent to the sphere.*

Demonstration. Let FAG be a plane perpendicular to the radius AO at its extremity; if we take any point M in this plane and join OM, AM, the angle OAM will be a right angle, and thus the distance OM will be greater than OA; consequently the point M is without the sphere; and, as the same might be shown with respect to every other point of the plane FAG, it follows that this plane has only the point A in common with the surface of the sphere; therefore it is a tangent to this surface (440).

470. *Scholium.* It may be shown, in like manner, that two spheres have only one point common, and are consequently tangents to each other, when the distance of their centres is equal to the sum or to the difference of their radii; in this case the centres and the point of contact are in the same straight line.

THEOREM.

471. *The angle* BAC (fig. 226), *which two arcs of great circles make with each other, is equal to the angle* FAG *formed by the tangents of these arcs at the point* A; *it has also for its measure the arc* DE, *described from the point* A *as a pole, and comprehended between the sides* AB, AC, *produced if necessary.* Fig. 226

Demonstration. For the tangent AF, drawn in the plane of the arc AB, is perpendicular to the radius AO (110); and the tangent AG, drawn in the plane of the arc AC, is perpendicular

to the same radius *AO*; therefore the angle *FAG* is equal to the angle of the planes *OAB*, *OAC* (349), which is that of the arcs *AB*, *AC*, and which is designated by *BAC*.

In like manner, if the arc *AD* is equal to a quadrant, and also *AE*, the lines *OD*, *OE*, will be perpendicular to *AO*, and the angle *DOE* will be equal to the angle of the planes *AOD*, *AOE*; therefore the arc *DE* is the measure of the angle of these planes, or the measure of the angle *CAB*.

472. *Corollary.* The angles of spherical triangles may be compared with each other by means of the arcs of great circles, described from their vertices as poles and comprehended between their sides; thus it is easy to make an angle equal to a given angle.

Fig. 228 473. *Scholium.* The angles opposite to each other at the vertex, as *ACO*, *BCN*, are equal; for each is equal to the angle formed by the two planes *ACB*, *OCN* (350).

It will be perceived also that in the meeting of two arcs *ACB*, *OCN*, the two adjacent angles *ACO*, *OCB*, taken together, are equal to two right angles.

THEOREM.

227

Fig. 207. 474. *The triangle* ABC (fig. 207) *being given, if from the points* A, B, C, *as poles, the arcs* EF, FD, DE, *be described forming the triangle* DEF, *reciprocally the points* D, E, F, *will be the poles of the sides* BC, AC, AB.

Demonstration. The point *A* being the pole of the arc *EF*, the distance *AE* is a quadrant; the point *C* being the pole of the arc *DE*, the distance *CE* is likewise a quadrant; consequently the point *E* is distant a quadrant from each of the points *A*, *C*; therefore it is the pole of the arc *AC* (467). It may be shown, in the same manner, that *D* is the pole of the arc *BC*, and *F* that of the arc *AB*.

475. *Corollary.* Hence the triangle *ABC* may be described by means of *DEF*, as *DEF* is described by means of *ABC*.

THEOREM.

Fig. 227. 476. *The same things being supposed as in the preceding theorem, each angle of one of the triangles* ABC, DEF (fig. 227), *will have for its measure a semicircumference minus the side opposite in the other triangle.*

Demonstration. Let the sides AB, AC, be produced, if necessary, till they meet EF in G and H; since the point A is the pole of the arc GH, the angle A will have for its measure the arc GH. But the arc EH is a quadrant, as also GF, since E is the pole of AH, and F the pole of AG (465); consequently $EH + GF$ is equal to a semicircumference. But $EH + GF$ is the same as $EF + GH$; therefore the arc GH, which measures the angle A, is equal to a semicircumference minus the side EF; likewise the angle B has for its measure $\frac{1}{2}$ *circ.* — DF, and the angle C $\frac{1}{2}$ *circ.* — DE.

This property must be reciprocal between the two triangles, since they are described in the same manner, the one by means of the other. Thus we shall find that the angles D, E, F, of the triangle DEF have for their measure respectively $\frac{1}{2}$ *circ.* — BC, $\frac{1}{2}$ *circ.* — AC, $\frac{1}{2}$ *circ.* — AB. Indeed, the angle D, for example, has for its measure the arc MI; but

$$MI + BC = MC + BI = \tfrac{1}{2}\ \text{circ.};$$

therefore the arc MI, the measure of the angle D, $= \frac{1}{2}$ *circ.* — BC, and so of the others.

477. *Scholium.* It may be remarked that, beside the triangle DEF, three others may be formed by the intersection of the three arcs DE, EF, DF. But the proposition applies only to the central triangle, which is distinguished from the three others by this, that the two angles A, D, are situated on the same side of BC, the two B, E, on the same side of AC, and the two C, F, on the same side of AB.

Different names are given to the triangles ABC, DEF; we shall call them *polar triangles*.

LEMMA.

478. *The triangle* ABC (fig. 229) *being given, if from the pole* A *and with the distance* AC *an arc of a small circle* DEC *be described, if also from the pole* B *and with the distance* BC *the arc* DFC *be described, and from the point* D *where the arcs* DEC, DFC, *cut each other, the arcs of great circles* AD, DB, *be drawn; we say that of the triangle* ADB *thus formed the parts will be equal to those of the triangle* ACB. Fig. 229.

Demonstration. For, by construction the side $AD = AC$, $DB = BC$, and AB is common; therefore the two triangles will

have the sides equal, each to each. We say moreover, that the angles opposite to the equal sides are equal.

Indeed, if the centre of the sphere be supposed in O, we can suppose a solid angle formed at the point O by the three plane angles AOB, AOC, BOC; we can suppose likewise a second solid angle formed by the three plane angles AOB, AOD, BOD. And since the sides of the triangle ABC are equal to those of the triangle ADB, it follows that the plane angles, which form one of the solid angles, are equal to the plane angles which form the other solid angle, each to each. But the planes of any two angles in the one solid have the same inclination to each other as the planes of the homologous angles in the other (359); consequently the angles of the spherical triangle DAB are equal to those of the triangle CAB, namely, $DAB = BAC$, $DBA = CBA$, and $ADB = ACB$; therefore the sides and the angles of the triangle ADB are equal to the sides and angles of the triangle ACB.

479. *Scholium.* The equality of these triangles does not depend upon an absolute equality, or equality by superposition, for it would be impossible to make them coincide by applying the one to the other, at least except they should happen to be isosceles. The equality then under consideration is that which we have already called equality by *symmetry*, and for this reason we shall call the triangles ACB, ADB, *symmetrical triangles.*

THEOREM.

480. *Two triangles situated on the same sphere, or on equal spheres, are equal in all their parts, when two sides and the included angle of the one are equal to two sides and the included angle of the other, each to each.*

Fig. 230. *Demonstration.* Let the side $AB = EF$ (*fig.* 230), the side $AC = EG$, and the angle $BAC = FEG$, the triangle EFG can be placed upon the triangle ABC, or upon the triangle symmetrical with it ABD, in the same manner as two plane triangles are applied, when they have two sides and the included angle of the one respectively equal to two sides and the included angle of the other (36). Therefore all the parts of the triangle EFG will be equal to those of the triangle ABC, that is, beside the three parts which were supposed equal we shall have the side $BC = FG$, the angle $ABC = EFG$, and the angle $ACB = EGF$.

THEOREM.

481. *Two triangles situated on the same sphere, or on equal spheres, are equal in all their parts, when a side and the two adjacent angles of the one are equal to a side and the two adjacent angles of the other, each to each.*

Demonstration. For one of these triangles may be applied to the other as has been done in the analogous case of plane triangles (38).

THEOREM.

482. *If two triangles situated on the same sphere, or on equal spheres, are equilateral with respect to each other, they will also be equiangular with respect to each other, and the equal angles will be opposite to equal sides.*

Demonstration. This proposition is manifest from the reasoning pursued in art. 478, by which it is shown that with three given sides AB, AC, BC, only two triangles can be constructed, differing as to the position of their parts, but equal as to the magnitude of these parts. Therefore two triangles, which are equilateral with respect to each other, are either absolutely equal, or at least equal by symmetry; in either case they are equiangular with respect to each other and the equal angles are opposite to equal sides.

THEOREM.

483. *In every isosceles spherical triangle the angles opposite to the equal sides are equal; and conversely, if two angles of a spherical triangle are equal, the triangle is isosceles.*

Demonstration. 1. Let AB be equal AC (*fig.* 231); we say that the angle C will be equal to the angle B. For, if from the vertex A the arc AD be drawn to the middle of the base, the two triangles ABD, ADC, will have the three sides of the one equal to the three sides of the other, each to each, namely, AD common, $BD = DC$, $AB = AC$; consequently, by the preceding theorem, the two triangles will have their homologous angles equal, therefore $B = C$. Fig. 231.

2. Let the angle B be equal to C; we say that AB will be equal to AC. For, if the side AB is not equal to AC, let AB be

the greater; take $BO = AC$, and join OC. The two sides BO, BC, are equal to the two AC, BC; and the angle OBC contained by the first is equal to the angle ACB contained by the second. Consequently the two triangles have their other parts equal (480), namely, $OCB = ABC$; but the angle ABC is, by hypothesis, equal to ACB; whence OCB is equal to ACB, which is impossible; AB then cannot be supposed unequal to AC; therefore the sides AB, AC, opposite to the equal angles B, C, are equal.

484. *Scholium.* It is evident, from the same demonstration, that the angle $BAD = DAC$, and the angle $BDA = ADC$. Consequently the two last are right angles; therefore, *the arc drawn from the vertex of an isosceles spherical triangle to the middle of the base, is perpendicular to this base, and divides the angle opposite into two equal parts.*

THEOREM.

Fig. 232. 485. *In any spherical triangle* ABC (fig. 232), *if the angle* A *is greater than the angle* B, *the side* BC *opposite to the angle* A *will be greater than the side* AC *opposite to the angle* B; *conversely, if the side* BC *is greater than* AC, *the angle* A *will be greater than the angle* B.

Demonstration. 1. Let the angle $A > B$; make the angle $BAD = B$, and we shall have $AD = DB$ (483); but

$$AD + DC > AC;$$

in the place of AD substitute DB, and we shall have $DB + DC$ or $BC > AC$.

2. If we suppose $BC > AC$, we say that the angle BAC will be greater ABC. For, if BAC were equal to ABC, we should have $BC = AC$; and if BAC were less than ABC, it would follow, according to what has just been demonstrated, that $BC < AC$, which is contrary to the supposition, therefore the angle BAC is greater than ABC.

THEOREM.

Fig 233. 486. *If the two sides* AB, AC (fig. 233), *of the spherical triangle* ABC *are equal to the two sides* DE, DF, *of the triangle* DEF *described upon an equal sphere, if at the same time the angle* A *is greater than the angle* D, *we say that the third side* BC *of the first triangle will be greater than the third side* EF *of the second.*

The demonstration of this proposition is entirely similar to that of art. 42.

THEOREM.

487. *If two triangles described upon the same sphere or upon equal spheres are equiangular with respect to each other, they will also be equilateral with respect to each other.*

Demonstration. Let A, B, be the two given triangles, P, Q, their polar triangles. Since the angles are equal in the triangles A, B, the sides will be equal in the polar triangles P, Q (476); but, since the triangles P, Q, are equilateral with respect to each other, they are also equiangular with respect to each other (482); and, the angles being equal in the triangles P, Q, it follows that the sides are equal in their polar triangles A, B. Therefore the triangles A, B, which are equiangular with respect to each other, are at the same time equilateral with respect to each other.

This proposition may be demonstrated without making use of polar triangles in the following manner.

Let ABC, DEF (*fig.* 234), be two triangles equiangular with respect to each other, having $A = D$, $B = E$, $C = F$; we say that the sides will be equal, namely, $AB = DE$, $AC = DF$, $BC = EF$. Fig. 234.

Produce AB, AC, making $AG = DE$, $AH = DF$; join GH, and produce the arcs BC, GH, till they meet in I and K.

The two sides AG, AH, are, by construction, equal to the two DF, DE, the included angle $GAH = BAC = EDF$, consequently the triangles AGH, DEF, are equal in all their parts (480); therefore $AGH = DEF = ABC$, and the angle

$$AHG = DFE = ACB.$$

In the triangles IBG, KBG, the side BG is common, and the angle $IBG = GBK$; and, since $IGB + BGK$ is equal to two right angles, as also $GBK + IBG$, it follows that $BGK = IBG$. Consequently the triangles IBG, GBK, are equal (481); therefore $IG = BK$, and $IB = GK$.

In like manner, since the angle $AHG = ACB$, the triangles ICH, HCK, have a side and the two adjacent angles of the one respectively equal to a side and the two adjacent angles of the other; consequently they are equal; therefore $IH = CK$, and $HK = IC$.

Now, if from the equals BK, IG, we take the equals CK, IH, the remainders BC, GH, will be equal. Besides the angle $BCA = AHG$, and the angle $ABC = AGH$. Whence the triangles ABC, AHG, have a side and the two adjacent angles of the one respectively equal to a side and the two adjacent angles of the other; consequently they are equal. But the triangle DEF is equal in all its parts to the triangle AHG; therefore it is also equal to the triangle ABC, and we shall have $AB = DE$, $AC = DF$, $BC = EF$; hence, if two spherical triangles are equiangular with respect to each other, the sides opposite to the equal angles will be equal.

488. *Scholium.* This proposition does not hold true with regard to plane triangles, in which from the equality of the angles we can only infer the proportionality of the sides. But it is easy to account for the difference in this respect between plane and spherical triangles. In the present proposition as well as those of articles 480, 481, 482, 486, which relate to a comparison of triangles, it is said expressly that the triangles are described upon the same sphere or upon equal spheres. Now similar arcs are proportional to their radii; consequently upon equal spheres two triangles cannot be similar without being equal. It is not therefore surprising that equality of angles should imply equality of sides.

It would be otherwise, if the triangles were described upon unequal spheres; then, the angles being equal, the triangles would be similar, and the homologous sides would be to each other as the radii of the spheres.

THEOREM.

489. *The sum of the angles of every spherical triangle is less than six and greater than two right angles.*

Demonstration. 1. Each angle of a spherical triangle is less than two right angles (*see the following scholium*); therefore the sum of the three angles is less than six right angles.

2. The measure of each angle of a spherical triangle is equal to the semicircumference minus the corresponding side of the polar triangle (476); therefore the sum of the three angles has for its measure three semicircumferences minus the sum of the sides of the polar triangle. Now this last sum is less than a

circumference (461); consequently, by subtracting it from three semicircumferences the remainder will be greater than a semicircumference, which is the measure of two right angles; therefore the sum of the three angles of a spherical triangle is greater than two right angles.

490. *Corollary* I. The sum of the angles of a spherical triangle is not constant like that of a plane triangle; it varies from two right angles to six, without the possibility of being equal to either limit. Thus, two angles being given, we cannot thence determine the third.

491. *Corollary* II. A spherical triangle may have two or three right angles, also two or three obtuse angles.

If the triangle *ABC* (*fig.* 235) has two right angles *B* and *C*, the vertex *A* will be the pole of the base *BC* (467); and the sides *AB*, *AC*, will be quadrants. Fig. 235

If the angle *A* also is a right angle, the triangle *ABC* will have all its angles right angles, and all its sides quadrants. The triangle having three right angles is contained eight times in the surface of the sphere; this is evident from figure 236, if we suppose the arc *MN* equal to a quadrant.

492. *Scholium.* We have supposed in all that precedes, conformably to the definition art. 442, that spherical triangles always have their sides less each than a semicircumference; then it follows that the angles are always less than two right angles. For the side *AB* (*fig.* 224) is less than a semicircumference as also *AC*; these arcs must both be produced in order to meet in *D*. Now the two angles *ABC*, *CBD*, taken together, are equal to two right angles; therefore the angle *ABC* is by itself less than two right angles. Fig. 224

We will remark, however, that there are spherical triangles of which certain sides are greater than a semicircumference, and certain angles greater than two right angles. For, if we produce the side *AC* till it becomes an entire circumference *ACE*, what remains, after taking from the surface of the hemisphere the triangle *ABC*, is a new triangle, which may also be designated by *ABC*, and the sides of which are *AB*, *BC*, *AEDC*. We see then, that the side *AEDC* is greater than the semicircumference *AED*; but, at the same time, the opposite angle *B* exceeds two right angles by the quantity *CBD*.

Besides, if we exclude from the definition triangles, the sides and angles of which are so great, it is because the resolution of them, or the determination of their parts, reduces itself always to that of triangles contained in the definition. Indeed, it will be readily perceived, that if we know the angles and sides of the triangle ABC, we shall know immediately the angles and sides of the triangle of the same name, which is the remainder of the surface of the hemisphere.

THEOREM.

Fig. 236. 493. *The lunary surface* AMBNA (fig. 236) *is to the surface of the sphere as the angle* MAN *of this surface is to four right angles, or as the arc* MN, *which measures this angle, is to the circumference.*

Demonstration. Let us suppose in the first place, that the arc MN is to the circumference $MNPQ$ in the ratio of two entire numbers, as 5 to 48, for example. The circumference $MNPQ$ may be divided into 48 equal parts, of which MN will contain 5; then joining the pole A and the points of division by as many quadrants, we shall have 48 triangles in the surface of the hemisphere $AMNPQ$, which will be equal among themselves, since they have all their parts equal. The entire sphere then will contain 96 of these partial triangles, and the lunary surface $AMBNA$ will contain 10 of them; therefore the lunary surface is to that of the sphere as 10 is to 96, or as 5 is to 48, that is, as the arc MN is to the circumference.

If the arc MN is not commensurable with the circumference, it may be shown by a course of reasoning, of which we have already had many examples, that the lunary surface is always to that of the sphere as the arc MN is to the circumference.

494. *Corollary* I. Two lunary surfaces are to each other as their respective angles.

495. *Corollary* II. We have already seen that the entire surface of the sphere is equal to eight triangles having each three right angles (491); consequently, if the area of one of these triangles be taken for unity, the surface of the sphere will be represented by eight. This being supposed, the lunary surface, of which the angle is A, will be expressed by $2A$, the angle A being estimated by taking the right angle for unity; for we have $2A : 8 :: A : 4$. Here are then two kinds of units;

one for angles, this is the right angle; the other for surfaces, this is the spherical triangle, of which all the angles are right angles and the sides quadrants.

496. *Scholium.* The spherical wedge comprehended by the planes AMB, ANB, is to the entire sphere, as the angle A is to four right angles. For the lunary surfaces being equal, the spherical wedges will also be equal; therefore two spherical wedges are to each other as the angles formed by the planes which comprehend them.

THEOREM.

497. *Two symmetrical spherical triangles are equal in surface.*

Demonstration. Let ABC, DEF (*fig.* 237), be two symmetrical triangles, that is two triangles which have their sides equal, namely, $AB = DE$, $AC = DF$, $CB = EF$, and which at the same time do not admit of being applied the one to the other; we say that the surface ABC is equal to the surface DEF. Fig. 2

Let P be the pole of the small circle which passes through the three points A, B, C*; from this point draw the equal arcs PA, PB, PC (464); at the point F make the angle $DFQ = ACP$, the arc $FQ = CP$, and join DQ, EQ.

The sides DF, FQ, are equal to the sides AC, CP; the angle $DFQ = ACP$; consequently the two triangles DFQ, ACP, are equal in all their parts (480); therefore the side $DQ = AP$, and the angle $DQF = APC$.

In the proposed triangles DFE, ABC, the angles DFE, ACB, opposite to the equal sides DE, AB, being equal (481), if we subtract from them the angles DFQ, ACP, equal, by construction, there will remain the angle QFE equal to PCB. Moreover the sides QF, FE, are equal to the sides PC, CB; consequently the two triangles FQE, CPB, are equal in all their parts; therefore the side $QE = PB$, and the angle $FQE = CPB$.

If we observe now that the triangles DFQ, ACP, which have the sides equal each to each, are at the same time isosceles, we

* The circle, which passes through the three points A, B, C, or which is circumscribed about the triangle ABC, can only be a small circle of the sphere; for, if it were a great one, the three sides AB, BC, AC, would be situated in the same plane, and the triangle ABC would be reduced to one of its sides.

shall perceive that they may be applied the one to the other; for, having placed PA upon its equal QF, the side PC will fall upon its equal QD, and thus the two triangles will coincide; consequently they are equal, and the surface $DQF = APC$. For a similar reason the surface $FQE = CPB$, and the surface $DQE = APB$; we have accordingly
$DQF + FQE - DQE = APC + CPB - APB$, or $DEF = ABC$; therefore the two symmetrical triangles ABC, DEF, are equal in surface.

498. *Scholium.* The poles P and Q may be situated within the triangles ABC, DEF; then it would be necessary to add the three triangles DQF, FQE, DQE, in order to obtain the triangle DEF, and also the three triangles APC, CPB, APB, in order to obtain the triangle ABC. In other respects the demonstration would always be the same and the conclusion the same.

THEOREM.

Fig. 238. 499. *If two great circles* AOB, COD (fig. 238), *cut each other in any manner in the surface of a hemisphere* AOCBD, *the sum of the opposite triangles* AOC, BOD, *will be equal to the lunary surface of which the angle is* BOD.

Demonstration. By producing the arcs OB, OD, into the surface of the other hemisphere till they meet in N, OBN will be a semicircumference as well as AOB; taking from each OB, we shall have $BN = AO$. For a similar reason $DN = CO$, and $BD = AC$; consequently the two triangles AOC, BDN, have the three sides of the one equal respectively to the three sides of the other; moreover, their position is such that they are symmetrical; therefore they are equal in surface (496), and the sum of the triangles AOC, BOD, is equivalent to the lunary surface $OBNDO$, of which the angle is BOD.

500. *Scholium.* It is evident also that the two spherical pyramids which have for their bases the triangles AOC, BOD, taken together, are equal to the spherical wedge of which the angle is BOD.

THEOREM.

501. *The surface of a spherical triangle has for its measure the excess of the sum of the three angles over two right angles.*

Demonstration. Let ABC (*fig.* 239) be the triangle proposed; Fig. 23
produce the sides till they meet the great circle $DEFG$ drawn at pleasure without the triangle. By the preceding theorem the two triangles ADE, AGH, taken together, are equal to the lunary surface of which the angle is A, and which has for its measure $2A$ (495); thus we shall have $ADE + AGH = 2A$; for a similar reason $BGF + BID = 2B$, $CIH + CFE = 2C$. But the sum of these six triangles exceeds the surface of a hemisphere by twice the triangle ABC; moreover the surface of a hemisphere is represented by 4; consequently the double of the triangle ABC is equal to $2A + 2B + 2C - 4$, and consequently $ABC = A + B + C - 2$; therefore every spherical triangle has for its measure the sum of its angles minus two right angles.

502. *Corollary* I. The proposed triangle will contain as many triangles of three right angles, or eighths of the sphere (494), as there are right angles in the measure of this triangle. If the angles, for example, are each equal to $\frac{4}{3}$ of a right angle, then the three angles will be equal to four right angles, and the proposed triangle will be represented by $4 - 2$ or 2; therefore it will be equal to two triangles of three right angles, or to a fourth of the surface of the sphere.

503. *Corollary* II. The spherical triangle ABC is equivalent to a lunary surface, the angle of which is $\frac{A + B + C}{2} - 1$; likewise the spherical pyramid, the base of which is ABC, is equal to the spherical wedge, the angle of which is $\frac{A + B + C}{2} - 1$.

504. *Scholium.* At the same time that we compare the spherical triangle ABC with the triangle of three right angles, the spherical pyramid, which has for its base ABC, is compared with the pyramid which has a triangle of three right angles for its base, and we obtain the same proportion in each case. The solid angle at the vertex of a pyramid is compared in like manner with the solid angle at the vertex of the pyramid having a triangle of three right angles for its base. Indeed the comparison is established by the coincidence of the parts. Now, if the bases of pyramids coincide, it is evident that the pyramids themselves will coincide, as also the solid angles at the vertex. Whence we derive several consequences;

1. Two spherical triangular pyramids are to each other as their bases; and, since a polygonal pyramid may be divided into several triangular pyramids, it follows that any two spherical pyramids are to each other as the polygons which constitute their bases.

2. The solid angles at the vertex of these same pyramids are likewise proportional to the bases; therefore, in order to compare any two solid angles, the vertices are to be placed at the centres of two equal spheres, and these solid angles will be to each other as the spherical polygons intercepted between their planes or faces.

The angle at the vertex of the pyramid, whose base is a triangle of three right angles is formed by three planes perpendicular to each other; this angle, which may be called a *solid right angle*, is very proper to be used as the unit of measure for other solid angles. This being supposed, the same number which gives the area of a spherical polygon will give the measure of the corresponding solid angle. If, for example, the area of a spherical polygon is $\frac{3}{4}$, that is, if it is $\frac{3}{4}$ of a triangle of three right angles, the corresponding solid angle will also be $\frac{3}{4}$ of a solid right angle.

THEOREM.

505. *The surface of a spherical polygon has for its measure the sum of its angles minus the product of two right angles by the number of sides in the polygon minus two.*

Fig. 240. *Demonstration.* From the same vertex A (*fig.* 240) let there be drawn to the other vertices the diagonals AC, AD; the polygon $ABCDE$ will be divided into as many triangles minus two as it has sides. But the surface of each triangle has for its measure the sum of its angles minus two right angles, and it is evident that the sum of all the angles of the triangles is equal to the sum of the angles of the polygon; therefore the surface of the polygon is equal to the sum of its angles diminished by as many times two right angles as there are sides minus two.

506. *Scholium.* Let s be the sum of the angles of a spherical polygon, n the number of its sides; the right angle being supposed unity, the surface of the polygon will have for its measure $s - 2(n - 2)$ or $s - 2n + 4$.

SECTION FOURTH.

Of the three round bodies.

DEFINITIONS.

507. We call a *cylinder* the solid generated by the revolution of a rectangle $ABCD$ (*fig.* 250), which may be conceived to turn about the side AB considered as fixed. Fig. 250

During this revolution the sides AD, BC, remaining always perpendicular to AB, describe equal circular planes DHP, CGQ, which are called the *bases* of the cylinder, and the side CD describes the *convex surface of the cylinder.*

The fixed line AB is called the *axis of the cylinder.*

Every section KLM made by a plane perpendicular to the axis, is a circle equal to each of the bases; for, while the rectangle $ABCD$ turns about AB, the line IK, perpendicular to AB, describes a circular plane equal to the base, and this plane is simply the section made perpendicular to the axis at the point I.

Every section $PQGH$, made by a plane passing through the axis, is a rectangle double of the generating rectangle $ABCD$.

508. We call a *cone* the solid generated by the revolution of a right-angled triangle SAB (*fig.* 251), which may be conceived to turn about the fixed side SA. Fig. 251

In this revolution the side AB describes a circular plane $BDCE$ called the *base of the cone,* and the hypothenuse SB describes the *convex surface of the cone.*

The point S is called the *vertex of the cone,* SA the *axis* or *altitude,* and SB the *side.*

Every section $HKFI$, made perpendicularly to the axis, is a circle; every section SDE, made through the axis, is an isosceles triangle double of the generating triangle SAB.

509. If from the cone $SCDB$ we separate by a section parallel to the base the cone $SFKH$, the remaining solid $CBHF$ is called a *truncated cone* or a *frustum of a cone.* It may be conceived to be generated by the revolution of the trapezoid $ABHG$, of which the angles A and G are right angles, about the side AG. The fixed line AG is called the axis or altitude of the frustum, the circles BDC, HKF, are the *bases* and BH the *side* of the frustum.

510. Two cylinders or two cones are *similar*, when their axes are to each other as the diameters of their bases.

Fig. 252. 511. If, in the circle *ACD* (*fig.* 252), considered as the base of a cylinder, a polygon *ABCDE* be inscribed, and upon the base *ABCDE* a right prism be erected equal in altitude to the cylinder, the prism is said to be *inscribed in the cylinder*, or the cylinder to be *circumscribed about the prism.*

It is manifest that the edges *AF*, *BG*, *CH*, &c., of the prism, being perpendicular to the plane of the base, are comprehended in the convex surface of the cylinder; therefore the prism and cylinder touch each other along these lines.

Fig. 253. 512. In like manner, if *ABCD* (*fig.* 253) be a polygon circumscribed about the base of a cylinder, and upon the base *ABCD* a right prism, equal in altitude to the cylinder, be constructed, the prism is said to be *circumscribed about the cylinder*, or the cylinder *inscribed in the prism.*

Let *M*, *N*, &c., be the points of contact of the sides *AB*, *BC*, &c., and through the points *M*, *N*, &c., let the lines *MX*, *NY*, &c., be drawn perpendicular to the plane of the base; it is evident that these perpendiculars will be in the surface of the cylinder and in that of the circumscribed prism at the same time; therefore they will be lines of contact.

N. B. The cylinder, the cone and the sphere are the *three round bodies*, which are treated of in the elements.

Preliminary lemmas upon surfaces.

Fig. 254. 513. I. *A plane surface* OABCD (fig. 254) *is less than any other surface* PABCD *terminated by the same perimeter* ABCD.

Demonstration. This proposition is sufficiently evident to be ranked among the number of axioms; for we may consider the plane among surfaces what the straight line is among lines. The straight line is the shortest distance between two given points; in like manner the plane is the least surface among all those which have the same perimeter. Still, as it is proper to make the number of axioms as small as possible, I shall present a process of reasoning which will leave no doubt with regard to this proposition.

As a surface is extension in length and breadth, we cannot conceive one surface to be greater than another, except the dimensions of the first exceed in some direction those of the second:

and, if it happens that the dimensions of one surface are in all directions less than the dimensions of another surface, it is evident that the first surface will be less than the second. Now, in whatever direction the plane *BPD* be made to pass, as it cuts the plane in *BD*, and the other surface in *BPD*, the straight line *BD* will always be less than *BPB*; therefore the plane urface *OABCD* is less than the surrounding surface *PABCD*.

514. II. *A* convex *surface* OABCD (*fig.* 255) *is less than any other surface which encloses it by resting on the same perimeter* ABCD. Fig. 255.

Demonstration. We repeat here, that we understand by a *convex surface* a surface that cannot be met by a straight line in more than two points; still it is possible that a straight line may apply itself exactly to a convex surface in a certain direction; we have examples of this in the surfaces of the cone and cylinder. It should be observed moreover, that the denomination of *convex surface* is not confined to curved surfaces; it comprehends *polyedral* faces, or surfaces composed of several planes, also surfaces that are in part curved and in part polyedral.

This being premised, if the surface *OABCD* is not smaller than any of those which enclose it, let there be among these last *PABCD* the smallest surface which shall be at most equal to *OABCD*. Through any point *O* suppose a plane to pass touching the surface *OABCD* without cutting it; this plane will meet the surface *PABCD*, and the part which it separates from it will be greater than the plane terminated by the same surface; therefore by preserving the rest of the surface *PABCD*, we can substitute the plane for the part taken away, and we shall have a new surface, which encloses the surface *OABCD*, and which would be less than *PABCD*. But this last is the least of all, by hypothesis; consequently this hypothesis cannot be maintained; therefore the convex surface *OABCD* is less than any which encloses *OABCD* and which is terminated by the same perimeter *ABCD*.

515. *Scholium.* By a course of reasoning entirely similar it may be shown,

1. That if a convex surface terminated by two perimeters, *ABC*, *DEF* (*fig.* 256), is enclosed by any other surface terminated by the same perimeters, the enclosed surface will be less than the other. Fig. 256.

Fig. 257. 2. That, if a convex surface AB (*fig.* 257) is enclosed on all sides by another surface MN, whether they have points, lines, or planes in common, or whether they have no point in common, the enclosed surface is always less than the enclosing surface.

For among these last there cannot be one which shall be the least of all, since in all cases we can draw the plane CD a tangent to the convex surface, which plane would be less than the surface CMD; and thus the surface CND would be smaller than MN, which is contrary to the hypothesis, that MN is the smallest of all. Therefore the convex surface AB is less than any which encloses it.

THEOREM.

516. *The solidity of a cylinder is equal to the product of its base by its altitude.*

Fig. 258. *Demonstration.* Let CA (*fig.* 258) be the radius of the base of the given cylinder, H its altitude; and let *surf.* CA represent the surface of a circle whose radius is CA; we say that the solidity of the cylinder will be *surf.* $CA \times H$. For, if *surf.* $CA \times H$ is not the measure of the given cylinder, this product will be the measure of a cylinder either greater or less. In the first place let us suppose that it is the measure of a less cylinder, of a cylinder, for example, of which CD is the radius of the base and H the altitude.

Circumscribe about the circle, of which CD is the radius, a regular polygon $GHIP$, the sides of which shall not meet the circumference of which CA is the radius (285); then suppose a right prism having for its base the polygon $GHIP$, and for its altitude H; this prism will be circumscribed about the cylinder of which the radius of the base is CD. This being premised, the solidity of the prism is equal to the product of its base $GHIP$ multiplied by the altitude H; and the base $GHIP$ is less than the circle whose radius is CA; therefore the solidity of the prism is less than *surf.* $CA \times H$. But *surf.* $CA \times H$ is, by hypothesis, the solidity of the cylinder inscribed in the prism; consequently the prism would be less than the cylinder; but the cylinder on the contrary is less than the prism, because it is contained in it; therefore it is impossible that *surf.* $CA \times H$ should be the measure of a cylinder of which the radius of the base is CD and the

altitude H; or in more general terms *the product of the base of a cylinder by its altitude cannot be the measure of a less cylinder.*

We say, in the second place, that this same product cannot be the measure of a greater cylinder; for, not to multiply figures, let CD be the radius of the base of the given cylinder; and, if it be possible, let *surf.* $CD \times H$ be the measure of a greater cylinder, of a cylinder, for example, of which CA is the radius of the base and H the altitude.

The same construction being supposed as in the first case, the prism circumscribed about the given cylinder will have for its measure $GHIP \times H$; the area $GHIP$ is greater than *surf.* CD; consequently, the solidity of the prism in question is greater than *surf.* $CD \times H$; the prism then would be greater than the cylinder of the same altitude whose base is *surf.* CA. But the prism on the contrary is less than the cylinder, since it is contained in it; therefore *it is impossible that the product of the base of a cylinder by its altitude should be the measure of a greater cylinder.*

We conclude then, that the solidity of a cylinder is equal to the product of its base by its altitude.

517. *Corollary* I. Cylinders of the same altitude are to each other as their bases, and cylinders of the same base are to each other as their altitudes.

518. *Corollary* II. Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of the bases. For the bases are as the squares of their diameters; and, since the cylinders are similar, the diameters of the bases are as the altitudes (510); consequently the bases are as the squares of the altitudes; therefore the bases multiplied by the altitudes, or the cylinders themselves, are as the cubes of the altitudes.

519. *Scholium.* Let R be the radius of the base of a cylinder, H its altitude, the surface of the base will be πR^2 (291), and the solidity of the cylinder will be $\pi R^2 \times H$, or $\pi R^2 H$.

LEMMA.

520. *The convex surface of a right prism is equal to the perimeter of its base multiplied by its altitude.*

Demonstration. This surface is equal to the sum of the rectangles $AFGB$, $BGHC$, $CHID$, &c. (*fig.* 252), which compose it. Fig. 252.

Now the altitudes AF, BG, CH, &c., of these rectangles are each equal to the altitude of the prism. Therefore the sum of these rectangles, or the convex surface of the prism, is equal to the perimeter of its base multiplied by its altitude.

521. *Corollary.* If two right prisms have the same altitude, the convex surfaces of these prisms will be to each other as the perimeters of the bases.

LEMMA.

522. *The convex surface of a cylinder is greater than the convex surface of any inscribed prism, and less than the convex surface of any circumscribed prism.*

Demonstration. The convex surface of the cylinder and that
Fig. 252. of the inscribed prism $ABCDEF$ (*fig.* 252) may be considered as having the same length, since every section made in the one and the other parallel to AF is equal to AF; and if, in order to obtain the magnitude of these surfaces, we suppose them to be cut by planes parallel to the base, or perpendicular to the edge AF, the sections will be equal, the one to the circumference of the base, and the other to the perimeter of the polygon $ABCDE$ less than this circumference; since therefore, the lengths being equal, the breadth of the cylindric surface is greater than that of the prismatic surface, it follows that the first surface is greater than the second.

By a course of reasoning entirely similar it may be shown that the convex surface of the cylinder is less than that of any
Fig. 253. circumscribed prism $BCDKLH$ (*fig.* 253).

THEOREM.

523. *The convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude.*

Fig. 258. *Demonstration.* Let CA (*fig.* 258) be the radius of the base of the given cylinder, H its altitude; and let *circ.* CA be the circumference of a circle whose radius is CA; we say that *circ.* $CA \times H$ will be the convex surface of the cylinder. For, if this proposition be denied, *circ.* $CA \times H$ must be the surface of a cylinder either greater or less; and, in the first place, let us suppose that it is the surface of a less cylinder, of a cylinder, for example, of which the radius of the base is CD and the altitude H.

Circumscribe about the circle, whose radius is CD, a regular polygon $GHIP$, the sides of which shall not meet the circumference whose radius is CA; then suppose a right prism, whose altitude is H, and whose base is the polygon $GHIP$. The convex surface of this prism will be equal to the perimeter of the polygon $GHIP$ multiplied by its altitude H (520); this perimeter is less than the circumference of the circle whose radius is CA; consequently the convex surface of the prism is less than *circ.* $CA \times H$. But *circ.* $CA \times H$ is, by hypothesis, the convex surface of a cylinder of which CD is the radius of the base, which cylinder is inscribed in the prism; whence the convex surface of the prism would be less than that of the inscribed cylinder. But on the contrary it is greater (522); accordingly the hypothesis with which we set out is absurd; therefore, 1. *the circumference of the base of a cylinder multiplied by its altitude cannot be the measure of the convex surface of a less cylinder.*

We say in the second place, that this same product cannot be the measure of the surface of a greater cylinder. For, not to change the figure, let CD be the radius of the base of the given cylinder, and, if it be possible, let *circ.* $CD \times H$ be the convex surface of a cylinder, which with the same altitude has for its base a greater circle, the circle, for example, whose radius is CA. The same construction being supposed as in the first hypothesis, the convex surface of the prism will always be equal to the perimeter of the polygon $GHIP$, multiplied by the altitude H. But this perimeter is greater than *circ.* CD; consequently the surface of the prism would be greater than *circ.* $CD \times H$, which, by hypothesis, is the surface of a cylinder of the same altitude of which CA is the radius of the base. Whence the surface of the prism would be greater than that of the cylinder. But while the prism is inscribed in the cylinder, its surface will be less than that of the cylinder (522); for a still stronger reason is it less when the prism does not extend to the cylinder; consequently the second hypothesis cannot be maintained; therefore, 2. *the circumference of the base of a cylinder multiplied by its altitude cannot be the measure of the surface of a greater cylinder.*

We conclude then that the convex surface of a cylinder is equal to the circumference of the base multiplied by its altitude.

THEOREM.

524. *The solidity of a cone is equal to the product of its base by a third part of its altitude.*

Fig. 259. *Demonstration.* Let SO (*fig.* 259) be the altitude of the given cone, AO the radius of the base; representing by *surf.* AO the surface of the base, we say that the solidity of the cone is equal to *surf.* $AO \times \frac{1}{3} SO$.

1. Let *surf.* $AO \times \frac{1}{3} SO$ be supposed to be the solidity of a greater cone, of a cone, for example, whose altitude is always SO, but of which BO, greater than AO, is the radius of the base.

About the circle, whose radius is AO, circumscribe a regular polygon $MNPT$, which shall not meet the circumference of which OB is the radius (285); suppose then a pyramid having this polygon for its base and the point S for its vertex. The solidity of this pyramid is equal to the area of the polygon $MNPT$ multiplied by a third of the altitude SO (416). But the polygon is greater than the inscribed circle represented by *surf.* AO; consequently, the pyramid is greater than

$$surf.\ AO \times \tfrac{1}{3} SO,$$

which, by hypothesis, is the measure of the cone of which S is the vertex, and OB the radius of the base. But on the contrary the pyramid is less than the cone, since it is contained in it; therefore it is impossible that the base of the cone multiplied by a third of its altitude should be the measure of a greater cone.

2. We say, moreover, that this same product cannot be the measure of a smaller cone. For, not to change the figure, let OB be the radius of the base of the given cone, and, if it be possible, let *surf.* $OB \times \frac{1}{3} SO$ be the solidity of a cone which has for its altitude SO, and for its base the circle of which AO is the radius. The same construction being supposed as above, the pyramid $SMNPT$ will have for its measure the area $MNPT$ multiplied by $\frac{1}{3} SO$. But the area $MNPT$ is less than *surf.* OB; consequently the pyramid will have a measure less than

$$surf.\ OB \times \tfrac{1}{3} SO,$$

and accordingly it would be less than the cone, of which AO is the radius of the base and SO the altitude. But on the contrary the pyramid is greater than the cone, since it contains it; therefore it is impossible that the base of a cone multiplied by a third of its altitude should be the measure of a less cone.

We conclude then, that the solidity of a cone is equal to the product of its base by a third of its altitude.

525. *Corollary.* A cone is a third of a cylinder of the same base and same altitude; whence it follows,

1. That cones of equal altitudes are to each other as their bases;

2. That cones of equal bases are to each other as their altitudes;

3. That similar cones are as the cubes of the diameters of their bases, or as the cubes of their altitudes.

526. *Scholium.* Let R be the radius of the base of a cone, H its altitude; the solidity of the cone will be $\pi R^2 \times \frac{1}{3} H$, or $\frac{1}{3} \pi R^2 H$.

THEOREM.

527. *The frustum of a cone* ADEB (fig. 260) *of which* OA, DP, *are the radii of the bases, and* PO *the altitude, has for its measure* $\frac{1}{3} \pi \times \mathrm{OP} \times (\overline{\mathrm{AO}}^2 + \overline{\mathrm{DP}}^2 + \mathrm{AO} \times \mathrm{DP})$. Fig. 260

Demonstration. Let $TFGH$ be a triangular pyramid of the same altitude as the cone SAB, and of which the base FGH is equivalent to the base of the cone. The two bases may be supposed to be placed upon the same plane; then the vertices S, T, will be at equal distances from the plane of the bases; and the plane EPD produced will be in the pyramid the section IKL. We say now, that this section IKL is equivalent to the base DE; for the bases AB, DE, are to each other as the squares of the radii AO, DP (287), or as the squares of the altitudes SO, SP; the triangles FGH, IKL, are to each other as the squares of these same altitudes (407); consequently the circles AB, DE, are to each other as the triangles FGH, IKL. But, by hypothesis, the triangle FGH, is equivalent to the circle AB; therefore the triangle IKL is equivalent to the circle DE.

Now the base AB multiplied by $\frac{1}{3} SO$ is the solidity of the cone SAB, and the base FGH multiplied by $\frac{1}{3} SO$ is that of the pyramid $TFGH$; the bases therefore being equivalent, the solidity of the pyramid is equal to that of the cone. For a similar reason the pyramid $TIKL$ is equivalent to the cone SDE; therefore the frustum of the cone $ADEB$ is equivalent to the frustum of the pyramid $FGHIKL$. But the base FGH, equivalent to the circle of which the radius is AO, has for its measure

$\pi \times \overline{AO}^2$; likewise the base $IKL = \pi \times \overline{DP}^2$, and the mean proportional between $\pi \times \overline{AO}^2$ and $\pi \times \overline{DP}^2$ is $\pi \times AO \times DP$; therefore the solidity of the frustum of a pyramid or that of the frustum of a cone has for its measure

$$\tfrac{1}{3} OP \times (\pi \times \overline{AO}^2 + \pi \times \overline{DP}^2 + \pi \times AO \times DP) \quad (422),$$

or

$$\tfrac{1}{3} \pi \times OP \times (\overline{AO}^2 + \overline{PD}^2 + AO \times DP.)$$

THEOREM.

528. *The convex surface of a cone is equal to the circumference of its base multiplied by half its altitude.*

Fig. 259. *Demonstration.* Let AO (*fig.* 259), be the radius of the base of the given cone, S its vertex, and SA its side; we say that the surface will be *circ.* $AO \times \frac{1}{2} SA$. For, if it be possible, let *circ.* $AO \times \frac{1}{2} SA$ be the surface of a cone which has S for its vertex, and for its base the circle described with a radius OB greater than AO.

Circumscribe about the small circle a regular polygon $MNPT$, the sides of which shall not meet the circumference of which OB is the radius; and let $SMNPT$ be a regular pyramid, which has for its base the polygon, and for its vertex the point S. The triangle SMN, one of those which compose the convex surface of the pyramid, has for its measure the base MN multiplied by half of the altitude SA, which is at the same time the side of the given cone; this altitude being equal in all the triangles SNP, SPQ, &c., it follows that the convex surface of the pyramid is equal to the perimeter $MNPTM$ multiplied by $\frac{1}{2} SA$. But the perimeter $MNPTM$ is greater than *circ.* AO; therefore the convex surface of the pyramid is greater than *circ.* $AO \times \frac{1}{2} SA$, and consequently greater than the convex surface of the cone which, with the same vertex S, has for its base the circle described with the radius OB. But on the contrary the convex surface of the cone is greater than that of the pyramid; for if we apply the base of the pyramid to the base of an equal pyramid, and the base of the cone to that of an equal cone; the surface of the two cones will enclose on all sides the surface of the two pyramids; consequently the first surface will be greater than the second (514), and therefore the surface of the cone is greater than that of the pyramid, which is comprehended within

it. The contrary would be the consequence of our hypothesis; accordingly this hypothesis cannot be maintained; therefore the circumference of the base of a cone multiplied by the half of its side cannot be the measure of the surface of a greater cone.

2. We say also, that this same product cannot be the measure of the surface of a less cone. For let BO be the radius of the base of the given cone, and, if it be possible, let *circ.* $BO \times \frac{1}{2} SB$ be the surface of a cone of which S is the vertex, and AO, less than OB, the radius of the base.

The same construction being supposed as above, the surface of the pyramid $SMNPT$ will always be equal to the perimeter $MNPT$ multiplied by $\frac{1}{2} SA$. Now the perimeter $MNPT$ is less than *circ.* BO, and SA is less than SB; therefore for this double reason the convex surface of the pyramid is less than

$$circ.\ BO \times \tfrac{1}{2} SB,$$

which, by hypothesis, is the surface of a cone of which AO is the radius of the base; consequently the surface of the pyramid would be less than that of the inscribed cone. But on the contrary it is greater; for by applying the base of the pyramid to that of an equal pyramid, and the base of the cone to that of an equal cone, the surface of the two pyramids will enclose that of the two cones, and consequently will be greater. Therefore it is impossible that the circumference of the base of a given cone multiplied by the half of its side should be the measure of the surface of a less cone.

We conclude then that the convex surface of a cone is equal to the circumference of the base multiplied by half of its side.

529. *Scholium.* Let L be the side of a cone, and R the radius of the base, the circumference of this base will be $2\pi R$, and the surface of the cone will have for its measure $2\pi R \times \frac{1}{2} L$, or πRL.

THEOREM.

530. *The convex surface of the frustum of a cone* ADEB (fig. 261) *is equal to its side* AD *multiplied by the half sum of the circumferences of the two bases* AB, DE. Fig. 261.

Demonstration. In the plane SAB, which passes through the axis SO, draw perpendicularly to SA the line AF, equal to the circumference which has for its radius AO; join SF, and draw DH parallel to AF.

On account of the similar triangles SAO, SDC,

$$AO : DC :: SA : SD;$$

and, on account of the similar triangles SAF, SDH,

$$AF : DH :: SA : SD;$$

whence $AF : DH :: AO : DC ::$ *circ.* AO : *circ.* DC (287).
But, by construction, $AF =$ *circ.* AO; consequently

$$DH = \text{circ. } DC.$$

This being premised, the triangle SAF, which has for its measure $AF \times \frac{1}{2} SA$, is equal to the surface of a cone SAB, which has for its measure *circ.* $AO \times \frac{1}{2} SA$. For a similar reason the triangle SDH is equal to the surface of the cone SDE. Whence the surface of the frustum $ADEB$ is equal to that of the trapezoid $ADHF$. This has for its measure $AD \times \left(\frac{AF + DH}{2}\right)$ (178).
Therefore the surface of the frustum of a cone $ADEB$ is equal to its side AD multiplied by the half sum of the circumferences of the two bases.

531. *Corollary.* Through the point I, the middle of AD, draw IKL parallel to AB, and IM parallel to AF; it may be shown as above that $IM =$ *circ.* IK. But the trapezoid (179)

$$ADHF = AD \times IM = AD \times \text{circ. } IK.$$

Hence we conclude further that *the surface of the frustum of a cone is equal to its side multiplied by the circumference of a section made at equal distances from the two bases.*

532. *Scholium.* If a line AD, situated entirely on the same side of the line OC and in the same plane, make a revolution about OC, the surface described by AD will have for its measure $AD \times \left(\frac{\text{circ. } AO + \text{circ. } DC}{2}\right)$, or $AD \times$ *circ.* IK; the lines AO, DC, IK, being perpendiculars let fall from the extremities and from the middle of the line AD upon the axis OC.

For, if we produce AD and OC till they meet in S, it is evident that the surface described by AD is that of the frustum of a cone, of which OA and DC are the radii of the bases, the entire cone having for its vertex the point S. Therefore this surface will have the measure stated.

This measure would always be correct, although the point D should fall upon S, which would give an entire cone, and also when the line AD is parallel to the axis, which would give a cylinder. In the first case DC would be nothing, in the second DC would be equal to AO and to IK.

LEMMA.

533. *Let* AB, BC, CD (fig. 262), *be several successive sides of a regular polygon,* O *its centre, and* OI *the radius of the inscribed circle; if we suppose the portion of the polygon* ABCD, *situated entirely on the same side of the diameter* FG, *to make a revolution about this diameter, the surface described by* ABCD *will have for its measure* MQ × *circ.* OI, MQ *being the altitude of this surface, or the part of the axis comprehended between the extreme perpendiculars* AM, DQ. Fig. 262.

Demonstration. The point I being the middle of AB, and IK being a perpendicular to the axis let fall from the point I, the surface described by AB will have for its measure $AB \times circ.\ IK$ (532). Draw AX parallel to the axis, the triangles ABX, OIK, will have their sides perpendicular each to each, namely, OI to AB, IK to AX, and OK to BX; consequently these triangles will be similar, and will give the proportion

$$AB : AX \text{ or } MN :: OI : IK :: circ.\ OI : circ.\ IK,$$

therefore $AB \times circ.\ IK = MN \times circ.\ OI$. Whence it will be perceived that the surface described by AB is equal to its altitude MN multiplied by the circumference of the inscribed circle. Likewise the surface described by $BC = NP \times circ.\ OI$, the surface described by $CD = PQ \times circ.\ OI$. Accordingly the surface described by the portion of the polygon $ABCD$ has for its measure $(MN + NP + PQ) \times circ.\ OI$, or $MQ \times circ.\ OI$; therefore this surface is equal to its altitude multiplied by the circumference of the inscribed circle.

534. *Corollary.* If the entire polygon has an even number of sides, and the axis FG passes through two opposite vertices F and G, the entire surface described by the revolution of the semipolygon $FACG$ will be equal to its axis FG multiplied by the circumference of the inscribed circle. This axis FG will be at the same time the diameter of the circumscribed circle.

THEOREM.

535. *The surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*

Demonstration. 1. We say that the diameter of a sphere multiplied by the circumference of a great circle cannot be the

measure of the surface of a greater sphere. For, if it be possible, let $AB \times circ.\ AC$ (*fig.* 263) be the surface of a sphere whose radius is CD.

Fig. 263.

About the circle, whose radius is CA, circumscribe a regular polygon of an even number of sides, which shall not meet the circumference of the circle whose radius is CD; let M and S be two opposite vertices of this polygon; and about the diameter MS let the semipolygon MPS be made to revolve. The surface described by this polygon will have for its measure

$$MS \times circ.\ AC \quad (534);$$

but MS is greater than AB; therefore the surface described by the polygon is greater than $AB \times circ.\ AC$, and consequently greater than the surface of the sphere whose radius is CD. On the contrary the surface of the sphere is greater than the surface described by the polygon, since the first encloses the second on all sides. Therefore the diameter of a sphere multiplied by the circumference of a great circle cannot be the measure of the surface of a greater sphere.

2. We say also, that this same product cannot be the measure of the surface of a less sphere. For, if it be possible, let

$$DE \times circ.\ CD$$

be the surface of a sphere whose radius is CA. The same construction being supposed as in the first case, the surface of the solid generated by the polygon will always be equal to

$$MS \times circ.\ AC.$$

But MS is less than DE, and $circ.\ AC$ less than $circ.\ CD$; therefore for these two reasons the surface of the solid generated by the polygon would be less than $DE \times circ.\ CD$, and consequently less than the surface of the sphere whose radius is AC. But on the contrary the surface described by the polygon is greater than the surface of the sphere whose radius is AC, since the first surface encloses the second; therefore the diameter of a sphere multiplied by the circumference of a great circle cannot be the measure of the surface of a less sphere.

We conclude then, that the surface of a sphere is equal to the diameter multiplied by the circumference of a great circle.

536. *Corollary.* The surface of a great circle is measured by multiplying its circumference by half of the radius or a fourth of the diameter; therefore *the surface of a sphere is four times that of a great circle.*

537. *Scholium.* The surface of a sphere being thus measured and compared with plane surfaces, it will be easy to obtain the absolute value of lunary surfaces and spherical triangles, the ratio of which to the entire surface of the sphere has already been determined.

In the first place, the lunary surface, whose angle is A, is to the surface of the sphere, as the angle A is to four right angles (493), or as the arc of a great circle, which measures the angle A, is to the circumference of this same great circle. But the surface of the sphere is equal to this circumference multiplied by the diameter; therefore the lunary surface is equal to the arc, which measures the angle of this surface, multiplied by the diameter.

In the second place, every spherical triangle is equivalent to a lunary surface whose angle is equal to half of the excess of the sum of its three angles over two right angles (503). Let P, Q, R, be the arcs of a great circle which measure the three angles of a spherical triangle; let C be the circumference of a great circle and D its diameter; the spherical triangle will be equivalent to the lunary surface whose angle has for its measure $\frac{P+Q+R-\frac{1}{2}C}{2}$, and consequently its surface will be $D \times \frac{P+Q+R-\frac{1}{2}C}{2}$.

Thus, in the case of the triangle of three right angles, each of the arcs P, Q, R, is equal to $\frac{1}{4}C$, and their sum is $\frac{3}{4}C$, the excess of this sum over $\frac{1}{2}C$ is $\frac{1}{4}C$, and the half of this excess is $\frac{1}{8}C$; therefore the surface of a triangle of three right angles $= \frac{1}{8}C \times D$, which is the eighth part of the whole surface of the sphere.

The measure of spherical polygons follows immediately from that of triangles, and it is moreover entirely determined by the proposition of art. 505, since the unit of measure, which is the triangle of three right angles, has just been estimated on a plane surface.

THEOREM.

538. *The surface of any spherical zone is equal to the altitude of this zone multiplied by the circumference of a great circle.*

Demonstration. Let EF (*fig.* 269) be any arc, either less or Fig. 2
greater than a quadrant, and let FG be drawn perpendicular to

the radius EC; we say that the zone with one base, described by the revolution of the arc EF about EC, will have for its measure $EG \times circ.\ EC$.

For let us suppose, in the first place, that this zone has a less measure, and, if it be possible, let this measure be equal to $EG \times circ.\ AC$. Inscribe in the arc EF a portion of a regular polygon $EMNOPF$, the sides of which shall not touch the circumference described with the radius CA, and let fall upon EM the perpendicular CI, the surface described by the polygon EMF, turning about EC, will have for its measure $EG \times circ.\ CI$ (533). This quantity is greater than $EG \times circ.\ AC$, which, by hypothesis, is the measure of the zone described by the arc EF. Consequently the surface described by the polygon $EMNOPF$ would be greater than the surface described by the circumscribed arc EF; but on the contrary this last surface is greater than the first, since it encloses it on all sides; therefore the measure of any spherical zone with one base cannot be less than the altitude of this zone multiplied by the circumference of a great circle.

We say, in the second place, that the measure of the same zone cannot be greater than the altitude of this zone multiplied by the circumference of a great circle. For, let us suppose that the zone in question is the one described by the arc AB about AC, and, if it be possible, let the *zone* AB be greater than

$$AD \times circ.\ AC.$$

The entire surface of the sphere composed of the two *zones* AB, BH, has for its measure $AH \times circ.\ AC$ (535), or

$$AD \times circ.\ AC + DH \times circ.\ AC;$$

if then the *zone* AB be greater than $AD \times circ.\ AC$, the *zone* BH must be less than $DH \times circ.\ AC$, which is contrary to the first part already demonstrated. Therefore the measure of a spherical zone with one base cannot be greater than the altitude of this zone multiplied by the circumference of a great circle.

It follows then that every spherical zone with one base has for its measure the altitude of this zone multiplied by the circumference of a great circle.

Let us now consider any zone of two bases described by the
Fig. 220. revolution of the arc FH (*fig.* 220) about the diameter DE, and let FO, HQ, be drawn perpendicular to this diameter. The zone described by the arc FH is the difference of the two zones

described by the arcs DH and DF; these have for their measure respectively $DQ \times circ.\ CD$ and $DO \times circ.\ CD$; therefore the zone described by FH has for its measure

$$(DQ - DO) \times circ.\ CD, \text{ or } OQ \times circ.\ CD.$$

We conclude then that every spherical zone with one or two bases has for its measure the altitude of this zone multiplied by the circumference of a great circle.

539. *Corollary.* Two zones are to each other as their altitudes, and any zone whatever is to the surface of the sphere as the altitude of this zone is to the diameter.

THEOREM.

540. *If the triangle* BAC (fig. 264, 265) *and the rectangle* BCEF *of the same base and same altitude turn simultaneously about the common base BC, the solid generated by the revolution of the triangle will be a third of the cylinder generated by the revolution of the rectangle.* Fig. 26 26

Demonstration. Let fall upon the axis the perpendicular AD (*fig.* 264); the cone generated by the triangle ABD is a third of the cylinder generated by the rectangle $AFBD$ (524); also the cone generated by the triangle ADC is a third of the cylinder generated by the rectangle $ADCE$; therefore the sum of the two cones, or the solid generated by ABC, is a third of the sum of the two cylinders, or of the cylinder generated by the rectangle $BCEF$. Fig. 26

If the perpendicular AD (*fig.* 265) fall without the triangle, the solid generated by ABC will be the difference of the cones generated ABD and ACD; but, at the same time, the cylinder generated by $BCEF$ will be the difference of the cylinders generated by $AFBD$, $AECD$. Therefore the solid generated by the revolution of the triangle will be always the third of the cylinder generated by the revolution of the rectangle of the same base and same altitude. Fig. 26

541. *Scholium.* The circle of which AD is the radius has for its surface $\pi \times \overline{AD}^2$; consequently $\pi \times \overline{AD}^2 \times BC$ is the measure of the cylinder generated by $BCEF$, and $\frac{1}{3}\pi \times \overline{AD}^2 \times BC$ is the measure of the solid generated by the triangle ABC.

PROBLEM.

g. 266. 542. *The triangle* CAB (fig. 266) *being supposed to make a revolution about the line* CD, *drawn at pleasure without the triangle through the vertex* C, *to find the measure of the solid thus generated.*

Solution. Produce the side AB until it meet the axis CD in D, and from the points A, B, let fall upon the axis the perpendiculars AM, BN.

The solid generated by the triangle CAD has for its measure $\frac{1}{3}\pi \times \overline{AM}^2 \times CD$ (540); the solid generated by the triangle CBD has for its measure $\frac{1}{3}\pi \times \overline{BN}^2 \times CD$; therefore the difference of these solids, or the solid generated by ABC, will have for its measure $\frac{1}{3}\pi \times (\overline{AM}^2 - \overline{BN}^2) \times CD$.

This expression will admit of another form. From the point I, the middle of AB, draw IK perpendicular to CD, and through the point B draw BO parallel to CD, we shall have

$$AM + BN = 2IK \quad (178),$$

and $AM - BN = AO$; consequently $(AM + BN) \times (AM - BN)$, or $\overline{AM}^2 - \overline{BN}^2$ (184), is equal to $2IK \times AO$. Accordingly the measure of the solid under consideration will also be expressed by $\frac{2}{3}\pi \times IK \times AO \times CD$. But, if the perpendicular CP be let fall upon AB, the triangles ABO, DCP will be similar, and will give the proportion $AO : CP :: AB : CD$; whence

$$AO \times CD = CP \times AB;$$

moreover $CP \times AB$ is double of the area of the triangle ABC; thus we have $AO \times CD = 2ABC$; consequently the solid generated by the triangle ABC has also for its measure

$$\tfrac{4}{3}\pi \times ABC \times KI,$$

or, since *circ.* KI is equal to $2\pi \times KI$, this same measure will be $ABC \times \frac{2}{3}$ *circ.* KI. Therefore, *the solid generated by the revolution of the triangle* ABC *has for its measure the area of this triangle multiplied by two thirds of the circumference described by the point* I *the middle of the base.*

543. *Corollary.* If the side $AC = CB$, the line CI will be perpendicular to AB, the area ABC will be equal to $AB \times \frac{1}{2} CI$, and the solidity $\frac{4}{3}\pi \times ABC \times IK$ will become $\frac{2}{3}\pi \times AB \times IK \times CI$. But the triangles ABO, CIK, are similar, and give the proportion $AB : BO$ or $MN :: \mathrm{CI} : \mathrm{IK}$; consequently

$$AB \times IK = MN \times CI;$$

therefore the solid generated by the isosceles triangle ABC will have for its measure $\frac{2}{3}\pi \times MN \times \overline{CI}^2$.

544. *Scholium.* The general solution seems to suppose that the line AB produced would meet the axis, but the results would not be the less true, if the line AB were parallel to the axis.

Indeed the cylinder generated by $AMNB$ (*fig.* 268) has for its measure $\pi \times \overline{AM}^2 \times MN$, the cone generated by ACM is equal to Fig. 268

$$\tfrac{1}{3}\pi \times \overline{AM}^2 \times CM,$$

and the cone generated by $BCN = \frac{1}{3}\pi \times \overline{AM}^2 \times CN$. Adding the two first solids together and subtracting the third from the sum, we have for the solid generated by ABC

$$\pi \times \overline{AM}^2 \times (MN + \tfrac{1}{3}CM - \tfrac{1}{3}CN);$$

and, since $CN - CM = MN$, this expression reduces itself to $\pi \times \overline{AM}^2 \times \frac{2}{3}MN$, or $\frac{2}{3}\pi \times \overline{CP}^2 \times MN$, which agrees with the results already found.

THEOREM.

545. *Let* AB, BC, CD (fig. 262), *be several successive sides of a regular polygon,* O *its centre,* OI *the radius of the inscribed circle; if we suppose the polygonal sector* AOD, *situated on the same side of the diameter* FG, *to make a revolution about this diameter, the solid generated will have for its measure* $\frac{2}{3}\pi \times \overline{OI}^2 \times MQ$, MQ *being the portion of the axis terminated by the extreme perpendiculars* AM, DQ. Fig. 262

Demonstration. Since the polygon is regular, all the triangles AOB, BOC, &c., are equal and isosceles. Now, by the corollary of the preceding proposition, the solid generated by the isosceles triangle AOB has for its measure $\frac{2}{3}\pi \times \overline{OI}^2 \times MN$, the solid generated by the triangle BOC has for its measure $\frac{2}{3}\pi \times \overline{OI}^2 \times NP$, and the solid generated by the triangle COD has for its measure $\frac{2}{3}\pi \times \overline{OI}^2 \times PQ$; therefore the sum of these solids, or the entire solid generated by the polygonal sector AOD, has for its measure $\frac{2}{3}\pi \times \overline{OI}^2 \times (MN + NP + PQ)$, or $\frac{2}{3}\pi \times \overline{OI}^2 \times MQ$.

THEOREM.

546. *Every spherical sector has for its measure the zone which serves as a base multiplied by a third of the radius, and the entire sphere has for its measure its surface multiplied by a third of the radius.*

Fig. 269. *Demonstration.* Let ABC (*fig.* 269) be the circular sector, which, by its revolution about AC, generates the spherical sector; the zone described by AB being $AD \times circ. AC$, or

$$2\pi \times AC \times AD \quad (538),$$

we say that the spherical sector will have for its measure this zone multiplied by $\frac{1}{3} AC$, or $\frac{2}{3}\pi \times \overline{AC}^2 \times AD$.

1. Let us suppose, if it be possible, that this quantity

$$\frac{2}{3}\pi \times \overline{AC}^2 \times AD$$

is the measure of a greater spherical sector, of the spherical sector, for example, generated by the circular sector ECF similar to ACB.

Inscribe in the arc EF a portion of a regular polygon $EMNF$ the sides of which shall not meet the arc AB, then suppose the polygonal sector $ENFC$ to turn about EC at the same time with the circular sector ECF. Let CI be the radius of a circle inscribed in the polygon, and let FG be drawn perpendicular to EC. The solid generated by the polygonal sector will have for its measure $\frac{2}{3}\pi \times \overline{CI}^2 \times EG$ (545); now CI is greater than AC, by construction, and EG is greater than AD; for, if we join AB, EF, the triangles EFG, ABD, which are similar, give the proportion $EG : AD :: FG : BD :: CF : CB$; therefore $EG > AD$.

For this double reason $\frac{2}{3}\pi \times \overline{CI}^2 \times EG$ is greater than

$$\frac{2}{3}\pi \times \overline{CA}^2 \times AD;$$

the first expression is the measure of the solid generated by the polygonal sector, the second is, by hypothesis, that of the spherical sector generated by the circular sector ECF; consequently the solid generated by the polygonal sector would be greater than the spherical sector generated by the circular sector. But on the contrary the solid in question is less than the spherical sector, since it is contained in it; accordingly the hypothesis

with which we set out cannot be maintained; therefore the zone or base of a spherical sector multiplied by a third of the radius cannot be the measure of a greater spherical sector.

2. We say that this same product cannot be the measure of a less spherical sector. For, let *CEF* be the circular sector which by its revolution generates the given spherical sector, and let us suppose, if it be possible, that $\frac{2}{3}\pi \times \overline{CE}^2 \times EG$ is the measure of a less spherical sector, of that, for example, generated by the circular sector *ACB*.

The preceding construction remaining the same, the solid generated by the polygonal sector will always have for its measure $\frac{2}{3}\pi \times \overline{CI}^2 \times EG$. But *CI* is less than *CE*; consequently the solid is less than $\frac{2}{3}\pi \times \overline{CE}^2 \times EG$, which, by hypothesis, is the measure of the spherical sector generated by the circular sector *ACB*. Therefore the solid generated by the polygonal sector would be less than the solid generated by the spherical sector; but on the contrary it is greater, since it contains it. Therefore it is impossible that the zone of a spherical sector multiplied by a third of the radius should be the measure of a less spherical sector.

We conclude then, that every spherical sector has for its measure the zone which answers as a base multiplied by a third of the radius.

A circular sector *ACB* may be increased till it becomes equal to a semicircle; then the spherical sector generated by its revolution is an entire sphere. Therefore *the solidity of a sphere is equal to its surface multiplied by a third of the radius.*

547. *Corollary.* The surfaces of spheres being as the squares of their radii, these surfaces multiplied by the radii are as the cubes of the radii. Therefore *the solidities of two spheres are as the cubes of their radii,* or *as the cubes of their diameters.*

548. *Scholium.* Let R be the radius of a sphere, its surface will be $4\pi R^2$, and its solidity $4\pi R^2 \times \frac{1}{3}R$, or $\frac{4}{3}\pi R^3$. If we call D the diameter, we shall have $R = \frac{1}{2}D$, and $R^3 = \frac{1}{8}D^3$; therefore the solidity will also be expressed by $\frac{4}{3}\pi \times \frac{1}{8}D^3$, or $\frac{1}{6}\pi D^3$.

THEOREM.

549. *The surface of a sphere is to the whole surface of the circumscribed cylinder, the bases being comprehended, as 2 is to 3; and the solidities of these two bodies are in the same ratio.*

Fig. 270. *Demonstration.* Let *MPNQ* (*fig.* 270) be a great circle of the sphere, *ABCD* the circumscribed square; if the semicircle *PMQ*, and the semisquare *PADQ*, be made to turn at the same time about the diameter *PQ*, the semicircle will generate the sphere, and the semisquare will generate the cylinder circumscribed about the sphere.

The altitude *AD* of the cylinder is equal to the diameter *PQ*, the base of the cylinder is equal to a great circle, since it has for a diameter *AB* equal to *MN*; consequently the convex surface of the cylinder is equal to the circumference of a great circle multiplied by its diameter (523). This measure is the same as that of the surface of the sphere (535); whence it follows *that the surface of the sphere is equal to the convex surface of the circumscribed cylinder.*

But the surface of the sphere is equal to four great circles; consequently the convex surface of the circumscribed cylinder is also equal to four great circles. If we add the two bases, which are equal to two great circles, the whole surface of the circumscribed cylinder will be equal to six great circles; therefore the surface of the sphere is to the whole surface of the circumscribed cylinder as 4 is to 6, or as 2 is to 3. This is the first part of the proposition which it was proposed to demonstrate.

In the second place, since the base of the circumscribed cylinder is equal to a great circle, and its altitude equal to the diameter, the solidity of the cylinder will be equal to a great circle multiplied by the diameter (516). But the solidity of the sphere is equal to four great circles multiplied by a third of the radius (546), which amounts to a great circle multiplied by $\frac{4}{3}$ of the radius, or $\frac{2}{3}$ of the diameter; therefore the sphere is to the circumscribed cylinder as 2 is to 3, and consequently the solidities of these two bodies are to each other as their surfaces.

550. *Scholium.* If a polyedron be supposed, all whose faces touch the sphere, this polyedron might be considered as composed of pyramids having the centre of the sphere for their common vertex, the bases being the several faces of the polye-

dron. Now it is evident that all these pyramids will have for their common altitude the radius of the sphere, so that each pyramid will be equal to a face of the polyedron, which serves as a base, multiplied by a third of the radius; therefore the entire polyedron will be equal to its surface multiplied by a third of the radius of the inscribed sphere.

It will be perceived by this that the solidities of polyedrons circumscribed about a sphere are to each other as the surfaces of these same polyedrons. Thus the property which we have demonstrated for the circumscribed cylinder is common to an infinite number of other bodies.

We might have remarked also that the surfaces of polygons circumscribed about a circle are to each other as their perimeters.

PROBLEM.

551. *The circular segment* BMD (fig. 271) *being supposed to* Fig. 271
revolve about a diameter exterior to this segment, to find the value of the solid generated.

Solution. Let fall upon the axis the perpendiculars BE, DF, and upon the chord BD the perpendicular CI, and draw the radii CB, CD.

The solid generated by the sector $BCA = \frac{2}{3}\pi \times \overline{CB}^2 \times AE$ (546); the solid generated by the sector $DCA = \frac{2}{3}\pi \times \overline{CB}^2 \times AF$; consequently the difference of these two solids, or the solid generated by the sector DCB, will be equal to

$$\tfrac{2}{3}\pi \times \overline{CB}^2 \times (AF - AE) = \tfrac{2}{3}\pi \times \overline{CB}^2 \times EF.$$

But the solid generated by the isosceles triangle DCB has for its measure $\frac{2}{3}\pi \times \overline{CI}^2 \times EF$ (543); consequently the solid generated by the segment $BMD = \frac{2}{3}\pi \times EF \times \left(\overline{CB}^2 - \overline{CI}^2\right)$. Now in the right-angled triangle CBI we have $\overline{CB}^2 - \overline{CI}^2 = \overline{BI}^2 = \frac{1}{4}\overline{BD}^2$; therefore the solid generated by the segment BMD has for its measure $\frac{2}{3}\pi \times EF \times \frac{1}{4}\overline{BD}^2$, or $\frac{1}{6}\pi \times \overline{BD}^2 \times EF$.

552. *Scholium.* The solid generated by the segment BMD is to the sphere whose diameter is BD, as $\frac{1}{6}\pi \times \overline{BD}^2 \times EF$ is to $\frac{1}{6}\pi \times \overline{BD}^3$, or $:: EF : BD$.

THEOREM.

553. *Every segment of a sphere, comprehended between two parallel planes, has for its measure the half sum of its bases multiplied by its altitude, plus the solidity of the sphere of which this same altitude is the diameter.*

Fig. 271. *Demonstration.* Let BE, DF (*fig.* 271), be the radii of the bases of the segment, EF its altitude, so that the segment may be formed by the revolution of the circular space $BMDFE$ about the axis FE. The solid generated by the segment BMD will be equal to $\frac{1}{6}\pi \times \overline{BD}^2 \times EF$ (552), the frustum of a cone generated by the trapezoid $BDFE$ will be equal to

$$\tfrac{1}{3}\pi \times EF \times (\overline{BE}^2 + \overline{DF}^2 + BE \times DF) \quad (527);$$

consequently the segment of the sphere which is the sum of these two solids $= \frac{1}{6}\pi \times EF \times (2\overline{BE}^2 + 2\overline{DF}^2 + 2BE \times DF + \overline{BD}^2)$. But by drawing BO parallel to EF, we shall have $DO = DF - BE$, $\overline{DO}^2 = \overline{DF}^2 - 2DF \times BE + \overline{BE}^2$ (182), and consequently $\overline{BD}^2 = \overline{BO}^2 + \overline{DO}^2 = \overline{EF}^2 + \overline{DF}^2 - 2DF \times BE + \overline{BE}^2$. Putting this value in the place of $\overline{BD}^2$ in the expression for the segment, and reducing it, we shall have for the solidity of the segment

$$\tfrac{1}{6}\pi \times EF \times (3\overline{BE}^2 + 3\overline{DF}^2 + \overline{EF}^2),$$

an expression which may be decomposed into two parts; the one $\frac{1}{6}\pi \times EF \times (3\overline{BE}^2 + 3\overline{DF}^2)$, or $EF \times \left(\frac{\pi \times \overline{BE}^2 + \pi \times \overline{DF}^2}{2}\right)$, is the half sum of the bases multiplied by the altitude; the other $\frac{1}{6}\pi \times \overline{EF}^3$ represents the sphere of which EF is the diameter (548); therefore the segment of the sphere &c.

554. *Corollary.* If one of the bases is nothing, the segment in question becomes a spherical segment having only one base; therefore *every spherical segment having only one base is equivalent to half of the cylinder of the same base and same altitude, plus the sphere of which this altitude is the diameter.*

General Scholium.

555. Let R be the radius of the base of a cylinder, H its altitude; the solidity of the cylinder will be $\pi R^2 \times H$, or $\pi R^2 H$.

Let R be the radius of the base of a cone, H its altitude; the solidity of the cone will be $\pi R^2 \times \frac{1}{3} H$, or $\frac{1}{3} \pi R^2 H$.

Let A, B, be the radii of the bases of the frustum of a cone, H its altitude, the solidity of the frustum will be

$$\frac{1}{3} \pi H(A^2 + B^2 + AB).$$

Let R be the radius of a sphere; its solidity will be $\frac{4}{3} \pi R^3$.

Let R be the radius of a spherical sector, H the altitude of the zone which answers as a base; the solidity of the sector will be $\frac{2}{3} \pi R^2 H$.

Let P, Q, be the two bases of a spherical segment, H its altitude, the solidity of this segment will be $\left(\frac{P+Q}{2}\right) \times H + \frac{1}{6} \pi H^3$.

If the spherical segment have only one base P, its solidity will be $\frac{1}{2} PH + \frac{1}{6} \pi H^3$.

NOTES.

I.

Upon certain names and definitions.

Some new expressions and definitions have been introduced into this work which tend to give to the language of geometry more exactness and precision. We proceed to give an account of these changes, and to propose certain others, which might fulfil more completely the same purposes.

In the ordinary definition of a *rectangular parallelogram* and of a *square*, it is said that the angles of these figures are right angles; it would be more exact to say that their angles are equal. For, to suppose that the four angles of a quadrilateral may be right angles, and also that these right angles are equal to each other, is to suppose propositions which require to be demonstrated. This inconvenience and several others of the same kind might be avoided, if, instead of putting the definitions as is usual, at the head of a section, we distribute them through the section each in the place where the proposition implied is demonstrated.

The word *parallelogram* according to its etymology signifies *parallel lines*; it answers not better to a figure of four sides than to one of six, eight, &c., the opposite sides of which are parallel. Likewise the word *parallelopiped* signifies *parallel planes*; it does not designate a solid of six faces any more than one of eight, ten, &c., of which the opposite ones are parallel. It seems then that the denominations of parallelogram and parallelopiped, which have besides the inconvenience of being very long, ought to be banished from geometry. We might substitute in their place those of *rhomb* and *rhomboid*, which are much more convenient, and preserve the name of *lozenge* to denote a quadrilateral the sides of which are equal.

The word *inclination* ought to be understood in the same sense as that of angle; each indicates the manner of being of two lines, or of two planes, which meet, or which produced would meet. The inclination of two lines is nothing, when the angle is nothing, that is, when

the lines are parallel or coincident. The inclination is greatest, when the angle is greatest, or when the two lines make with each other a very obtuse angle. The quality of *leaning* is taken in a different sense; a line *leans* so much the more with respect to another, as it departs more from a perpendicular to this last.

The denomination of *equal triangles* is given by Euclid and others to those triangles, which are only equal in surface; and that of *equal solids* to those which are only equal in solidity. It appears to us more proper to call the triangles as well as the solids in this case *equivalent*, and to restrict the denomination of *equal triangles* and *equal solids* to those which would coincide upon being applied.

It is moreover necessary to distinguish among solids and curved surfaces two different kinds of equality. Indeed two solids, two solid angles, two spherical triangles, or two spherical polygons, may be equal in all their constituent parts without coinciding when applied. It does not appear that this observation has been made in elementary books; and, for want of having regard to it, certain demonstrations, founded upon the coincidence of figures, are not exact. Such are the demonstrations by which several authors pretend to prove the equality of spherical triangles in the same cases and in the same manner as they do that of plane triangles. We are furnished with a striking example of this by Robert Simson, who, in attacking the demonstration of the 28th proposition of the eleventh book of Euclid, fell himself into the error of founding his demonstration upon a coincidence which does not exist. We have thought it proper therefore to give a particular name to this kind of equality, which does not admit of coincidence; we have called it *equality by symmetry*; and the figures which are thus related we call *symmetrical* figures.

Thus the denominations of *equal* figures, *symmetrical* figures, *equivalent* figures, refer to different things and ought not to be confounded.

In the propositions, which relate to polygons, solid angles, and polyedrons, we have expressly excluded those which have re-entering angles. For, in addition to the advantage of considering in the elements only the most simple figures, if we had not thus restricted ourselves, certain propositions would either not have been true, or would have required to be modified. We have therefore confined ourselves to the consideration of lines and surfaces, which we call *convex*, and which are such that they cannot be cut by a straight line in more than two points.

We have often used the expression *product of two or of a greater number of lines*, by which we mean the product of the numbers

which represent these lines, they being estimated according to a linear unit taken at pleasure. The sense of this word being thus fixed, there is no difficulty in making use of it. The same is to be understood of the product of a surface by a line, of a surface by a solid, &c. It is sufficient to have established once for all that these products are or ought to be considered as the products of numbers, each of a kind that is adapted to it. Thus the product of a surface by a solid is nothing else than the product of a number of superficial units by a number of solid units.

We often use the word *angle* in common discourse to designate the point situated at its vertex; this expression is faulty. It would be more clear and more exact to denote by a particular name, as that of *vertices*, the points situated at the vertices of the angles of a polygon or of a polyedron. In this sense is to be understood the expression *vertices of a polyedron*, which we have used.

We have followed the common definition of *similar rectilineal figures*; but we would observe that it contains three superfluous conditions. For, in order to construct a polygon of which the number of sides is n, it is necessary in the first place to know a side, and then to have the position of the vertices of the angles situated without this side. Now the number of these angles is $n-2$, and the position of each vertex requires two data; whence it follows that the whole number of data necessary to construct a polygon of n sides is $1+2n-4$, or $2n-3$. But in the similar polygon there is one side to be taken at pleasure; thus the number of conditions, by which one polygon becomes similar to a given polygon, is $2n-4$. But the common definition requires, 1. that the angles should be equal, each to each, which makes n conditions; 2. that the homologous sides should be proportional, which makes $n-1$ conditions. There are then in all $2n-1$ conditions, or three too many. In order to obviate this inconvenience we can resolve the definition into two others, in this manner,

1. *Two triangles are similar, when they have two angles equal, each to each.*

2. *Two polygons are similar, when there can be formed in the one and the other the same number of triangles similar, each to each, and similarly disposed.*

But, in order that this last definition should not itself contain superfluous conditions, it is necessary that the number of triangles should be equal to the number of sides of the polygon minus two, which may take place in two ways. We can draw from two homologous angles diagonals to the opposite angles; then all the triangles formed in

each polygon will have a common vertex, and their sum will be equal to the polygon; or rather we can suppose that all the triangles formed in a polygon have for a common base a side of the polygon, and for vertices those of the different angles opposite to this base. In each case the number of triangles formed being $n-2$, the conditions of their similitude will be equal to the number $2n-4$; and the definition will contain nothing superfluous. This new definition being adopted, the ancient one will become a theorem, which may be demonstrated immediately.

If the definition of similar rectilineal figures is imperfect in books of elements, that of *similar solid polyedrons* is still more so. In Euclid this definition depends upon a theorem not demonstrated; in other authors it has the inconvenience of being very redundant; we have therefore rejected these definitions of similar solids†.

The definition of a *perpendicular to a plane* may be regarded as a theorem; that of the *inclination of two planes* also requires to be supported by reasoning; the same may be said of several others. It is on this account that, while we have placed the definitions according to ancient usage, we have taken care to refer to propositions where they are demonstrated; sometimes we have merely added a brief explanation which appeared sufficient.

The *angle* formed by the *meeting of two planes*, and the solid angle formed by the meeting of several planes in the same point, are distinct kinds of magnitudes, to which it would be well perhaps to give particular names. Without this it is difficult to avoid obscurity and circumlocutions in speaking of the arrangement of planes which compose the surface of a polyedron; and as the theory of solids has been little cultivated hitherto, there is less inconvenience in introducing new expressions, where they are required by the nature of the subject.

I should propose to give the name of *wedge* to the angle formed by two planes; the *edge* or *height* of the wedge would be the common intersection of the two planes. The wedge would be designated by four letters, of which the two middle ones would answer to the edge. A *right wedge* then would be the angle formed by two planes perpendicular to each other. Four right wedges would fill all the solid angular space about a given line. This new denomination would not prevent the wedge always having for its measure the angle formed by two lines drawn from the same point, the one in one of the

† The author here refers to a distinct note on the equality and similitude of polyedrons not given in this translation.

planes and the other in the other, perpendicularly to the edge or common intersection.

II.

The improvements referred to in the preceding note, so far as they have been adopted by the author, have been carefully preserved in the translation. Indeed it has been found necessary in a few instances to use English words in a sense somewhat different from their ordinary acceptation. The word *polygon* is generally restricted to figures of more than four sides. It is used in this work with the latitude of the original word *polygone* to stand for rectilineal figures generally; and *polyedron* is adopted in a similar manner for solids. *Quadrilateral* is employed as a general name for four-sided figures. The word *losenge* is rendered by *rhombus*, and *trapeze* by *trapezoid*, the English words, as they are commonly used, corresponding to the French. The perpendicular let fall from the centre of a regular polygon upon one of its sides is called in the original *apothéme*. It occurs but a few times, and as there is no English word answering to it, it is rendered by a periphrasis, or simply by the word perpendicular. The portion of the surface of a sphere comprehended between the semicircumferences of two great circles is denoted in the original by *fuseau*; Dr. Hutton uses the word *lune* in the same sense; others have employed *lunary surface*; as lune properly stands for the surface comprehended between two unequal circles, the latter denomination was thought the least exceptionable, and is adopted in the translation.

III.

On the demonstration of the proposition of article 58.

The proposition of art. 58 is only a particular case of the celebrated *postulate* upon which Euclid has established the theory of parallel lines, as well as the theorem upon the sum of the three angles of a triangle. This *postulate* has not yet been demonstrated in a manner entirely geometrical, and independent of the consideration of infinity, which is undoubtedly to be attributed to the imperfection of the definition of a straight line, which serves as the basis of the elements. But, if we consider this subject in a point of view more abstract, analysis offers a very simple method of demonstrating the proposition rigorously.

We show immediately by superposition, and without any preliminary proposition, that *two triangles are equal, when a side and the two adjacent angles of the one are equal* to a side and the two

adjacent angles of the other, each to each. **Let us call p the side in question, A and B the two adjacent angles, C the third angle. The angle C then must be entirely determinate, when the angles A and B are known with the side p; for, if several angles C could correspond to the three given things A, B, p, there would be as many different triangles, which would have a side and the two adjacent angles of the one equal to a side and the two adjacent angles of the other, which is impossible; therefore the angle C must be a determinate function of the three quantities A, B, p; which may be expressed thus**

$$C = \varphi : (A, B, p).$$

Let the right angle be equal to unity, then the angles A, B, C, will be numbers comprehended between 0 and 2; and, since

$$C = \phi\,(A, B, p),$$

we say that the line p does not enter into the function φ. Indeed we have seen that C must be entirely determined by the data A, B, p, merely, without any other angle or line whatever; but the line p is of a nature heterogeneous to the numbers A, B, C; and if, having any equation whatever among A, B, C, p, we could deduce the value of p in A, B, C, it would follow that p is equal to a number, which is absurd; therefore p cannot enter into the function φ, and we have simply $C = \phi : (A, B)$.....*.

This formula proves already that, if two angles of a triangle are equal to two angles of another triangle, the third must be equal to the third; and, this being supposed, it is easy to arrive at the theorem we have in view.

Fig. 274. In the first place let ABC (*fig.* 274) be a triangle right-angled at A; from the point A let fall upon the hypothenuse the perpendicular AD. The angles B and D of the triangle ABD are equal to the angles

* It has been objected to this demonstration that, if it were applied, word for word, to spherical trangles, it would follow that two known angles would be sufficient to determine the third, which would not be true in this kind of triangles. The answer is, that in spherical triangles there is one element more than in plane triangles, and this element is the radius of the sphere which must not be omitted. Accordingly, let r be the radius; then, instead of having $C = \phi\,(A, B, p)$, we shall have $C = \phi\,(A, B, p, r)$, or simply $C = \varphi\left(A, B, \frac{p}{r}\right)$, by the law of homogenious quantities. Now, since the ratio $\frac{p}{r}$ is a number, as well as A, B, C, there is nothing to prevent $\frac{p}{r}$ being found in the function φ, and then we can no longer conclude that

$$C = \varphi\,(A, B).$$

B and A of the triangle BAC; therefore, according to what has just been demonstrated, the third angle BAD is equal to the third C; for the same reason the angle $DAC = B$; consequently

$$BAD + DAC,$$

or $BAC = B + C$; but the angle BAC is a right angle; therefore *the two acute angles of a right-angled triangle, taken together, are equal to a right angle.*

Again, let BAC (*fig.* 275) be any triangle, and BC a side which is not less than each of the two others; if from the opposite angle A the perpendicular AD be let fall upon BC, this perpendicular will fall within the triangle ABC, and will divide it into two right-angled triangles BAD, DAC. Now in the right-angled triangle BAD the two angles BAD, ABD, are together equal to a right angle; in the right-angled triangle DAC the two angles DAC, ACD, are also equal to a right angle. Consequently the four united, or the three BAC, ABC, ACB, are together equal to two right angles; therefore *in every triangle the sum of the three angles is equal to two right angles.* Fig. 27

We see by this that the theorem, considered *a priori*, does not depend upon a series of propositions, but is deduced immediately from the principle of homogeniety, a principle which exists in every relation among quantities of whatever kind. But we proceed to show that another fundamental theorem of geometry may be deduced from the same source.

The above denominations being preserved, and the side opposite to the angle A being called m, and the side opposite the angle B being called n; the quantity m must be entirely determined by the quantities A, B, p; consequently m is a function of A, B, p, as also $\frac{m}{p}$ so that we can make $\frac{m}{p} = \psi : (A, B, p)$. But $\frac{m}{p}$ is a number, as well as A and B; therefore the function ψ must not contain the line p, and we have simply $\frac{m}{p} = \psi : (A, B)$, or $m = p\psi : (A, B)$. We have also in a similar manner $n = p\psi : (A, B)$.

Let there be another triangle formed with the same angles A, B, C, and having for the opposite sides m', n', p', respectively. Since A and B do not change, we have in this new triangle

$$m' = p'\psi\ (A, B),$$

and $n' = p'\psi : (B, A)$. Therefore $m : m' :: n : n' :: p : p'$. Therefore *in equiangular triangles the sides opposite to the equal angles are proportional.*

From this general proposition we deduce, as a particular case, that which we have supposed in the text for the demonstration of the
Fig. 35. proposition of art. 58. Indeed the triangles AFG, AML (*fig.* 35), have two angles equal, each to each, namely, the angle A common, and a right angle. Consequently these triangles are equiangular; therefore we have the proportion $AF : AL :: AG : AM$, by means of which the proposition is fully demonstrated†.

† In the note, of which the above is only a part, the author undertakes to demonstrate in a similar manner other fundamental propositions of geometry. For remarks upon the kind of reasoning here employed, the reader is referred to Leslie's Geometry, third edition, page 292.

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ELEMENTS

OF

ALGEBRA,

BY

S. F. LACROIX.

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ADVERTISEMENT.

LACROIX's Algebra has been in use in the French schools for a considerable time. It has been approved by the best judges, and been generally preferred to the other elementary treatises, which abound in France. The following translation is from the eleventh edition, printed at Paris in 1815. No alteration has been made from the original, except to substitute English instead of French measures in the questions, where it was thought necessary. When there has been an occasion to add a note of illustration, the reference is made by a letter or an obelisk, the author's being always distinguished by an asterisk.

Cambridge, June, 1818.

CONTENTS.

ELEMENTS OF ALGEBRA.

Preliminary Remarks upon the Transition from Arithmetic to Algebra—Explanation and Use of Algebraic Signs.

1. It must have been remarked in the *Elementary Treatise of Arithmetic,* that there are many questions, the solution of which is composed of two parts; the one having for its object to find to which of the four fundamental rules the determination of the unknown number by means of the numbers given belongs, and the other the application of these rules. The first part, (independent of the manner of writing numbers, or of the system of notation,) consists entirely in the development of the consequences which result (directly or indirectly) from the enunciation, or (from the manner in which that which is enunciated connects the numbers given with the numbers required, that is to say,) from the relations which it establishes between these numbers. If these relations are not complicated, we can for the most part find (by simple reasoning the value of the unknown numbers. In order to this it is necessary to analyze the conditions, which are involved in *the relations enunciated,* by reducing them to a course of equivalent expressions, of which the last ought to be one of the following; *the unknown quantity equal to the sum, or the difference, or the product, or the quotient, of such and such magnitudes.* This will be rendered plainer by an example.

To divide a given number into two such parts, that the first shall exceed the second by a given difference.

In order to this we would observe 1, that,

The greater part is equal to the less added to the given excess, and that by consequence, if the less be known, by adding to it this excess we have the greater; 2, that,

The greater added to the less forms the number to be divided.

Substituting in this last proposition, instead of the words, *the greater part*, the equivalent expression given above, namely, *the less part added to the given excess*, we find that,

The less part, added to the given excess, added moreover to the less part, forms the number to be divided.

But the language may be abridged, thus,

Twice the less part, added to the given excess, forms the number to be divided;

whence we infer, that,

Twice the less part is equal to the number to be divided diminished by the given excess;

and that,

Once the less part is equal to half the difference between the number to be divided and the given excess.

Or, which is the same thing,

The less part is equal to half the number to be divided, diminished by half the given excess.

The proposed question then is resolved, since to obtain the parts sought it is sufficient to perform operations purely arithmetical upon the given numbers.

If, for example, the number to be divided were 9, and the excess of the greater above the less 5, the less part would be according to the above rule, equal to $\frac{9}{2}$ less $\frac{5}{2}$, or $\frac{4}{2}$, or 2; and the greater, being composed of the less plus the excess 5, would be equal to 7.

2. The reasoning, which is so simple in the above problem, but which becomes very complicated in others, consists in general of a certain number of expressions, such as *added to*, *diminished by*, *is equal to*, &c. often repeated. These expressions relate to the operations by which the magnitudes, that enter into the enunciation of the question, are connected among themselves, and it is evident, that the expressions might be abridged by representing each of them by a sign. This is done in the following manner.

To denote addition we use the sign +, which signifies *plus*.

For subtraction we use the sign —, which signifies *minus*.

For multiplication we use the sign ×, which signifies *multiplied by*.

To denote that two quantities are to be divided one by the

other, we place the second under the first with a straight line between them; $\frac{5}{4}$ signifies 5 *divided by* 4.

Lastly, to indicate that two quantities are equal, we place between them the sign $=$ which signifies *equal*.

These abbreviations, although very considerable, are still not sufficient, for we are obliged often to repeat *the number to be divided, the number given, the less part, the number sought*, &c. by which the process is very much retarded.

With respect to given quantities, the expedient which first offers itself is, to take for representing them determinate numbers, as in arithmetic, but this not being possible with respect to the unknown quantities, the practice has been to substitute in their stead a conventional sign, which varies as occasion requires. We have agreed to employ the letters of the alphabet, generally using the last; as in arithmetic we put x for the fourth term of a proportion, of which only the three first are known. It is from the use of these several signs that we derive the science of *Algebra*.

I now proceed by means of them to consider the question stated above (1). I shall represent the unknown quantity, or the less number, by the letter x, for example, the number to be divided and the given excess by the two numbers 9 and 5; the greater number, which is sought, will be expressed by $x + 5$, and the sum of the greater and less by $x + 5 + x$; we have then

$$x + 5 + x = 9;$$

but by writing $2x$ for twice the quantity x there will result

$$2x + 5 = 9.$$

This expression shows that 5 must be added to the number $2x$ to make 9, whence we conclude that

$$2x = 9 - 5,$$

or that $$2x = 4,$$

and that lastly $$x = \frac{4}{2} = 2.$$

By comparing now the import of these abridged expressions, which I have just given by means of the usual signs, with the process of simple reasoning, by which we are lead to the solution, we shall see that the one is only a translation of the other.

The number 2, the result of the preceding operations, will answer only for the particular example which is selected, while the course of reasoning considered by itself, by teaching us,

that *the less part is equal to half the number to be divided, minus half the given excess*, renders it evident, that the unknown number is composed of the numbers given, and furnishes a rule by the aid of which we can resolve all the particular cases comprehended in the question.

The superiority of this method consists in its having reference to no one number in particular; the numbers given are used throughout without any change in the language by which they are expressed; whereas, by considering the numbers as determinate, we perform upon them, as we proceed, all the operations which are represented, and when we have come to the result there is nothing to show, how the number 2, to which we may arrive by any number of different operations, has been formed from the given numbers 9 and 5.

3. These inconveniences are avoided by using characters to represent the number to be divided and the given excess, that are independent of every particular value, and with which we can therefore perform any calculation. The letters of the alphabet are well adapted to this purpose, and the proposed question by means of them may be enunciated thus,

To divide a given number represented by a *into two such parts that the greater shall have with respect to the less a given excess represented by* b.

Denoting always the less by x;

The greater will be expressed by $x + b$;

Their sum, or the number to be divided, will be equal to $x + x + b$, or $2x + b$;

The first condition of the question then will give

$$2x + b = a.$$

Now it is manifest that, if it is necessary to add to double of x, or to $2x$, the quantity b in order to make the quantity a, it will follow from this, that it is necessary to diminish a by b to obtain $2x$, and that consequently $2x = a - b$.

We conclude then that half of $2x$ or $x = \frac{a}{2} - \frac{b}{2}$.

This last result, being translated into ordinary language, by substituting the words and phrases denoted by the letters and signs which it contains, gives the rule found before, according to which, *in order to obtain the less of two parts sought we sub-*

tract from half of the number to be divided, or from $\frac{a}{2}$ *half of the given excess, or* $\frac{b}{2}$.

Knowing the less part we have the greater by adding to the less the given excess. This remark is sufficient for effecting the solution of the question proposed; but Algebra does more; it furnishes a rule for calculating the greater part without the aid of the less as follows;

$\frac{a}{2}-\frac{b}{2}$ being the value of this, augmenting it by the excess b, we have for the greater part $\frac{a}{2}-\frac{b}{2}+b$. Now $\frac{a}{2}-\frac{b}{2}+b$ shows that after having subtracted from $\frac{a}{2}$ the half of b, it is necessary to add to the remainder the whole of b, or two halves of b, which reduces itself to augmenting $\frac{a}{2}$ by the half of b, or by $\frac{b}{2}$. It is evident then that $\frac{a}{2}-\frac{b}{2}+b$ becomes $\frac{a}{2}+\frac{b}{2}$; and by translating this expression we learn, that *of the two parts sought the greater is equal to half of the number to be divided plus half of the given excess.*

In the particular question which I first considered, the number to be divided was 9, the excess of one part above the other 5; in order to resolve it by the rules to which we have just arrived, it will be necessary to perform upon the numbers 9 and 5, the operations indicated upon a and b.

The half of 9 being $\frac{9}{2}$ and that of 5 being $\frac{5}{2}$, we have for the less part,

$$\tfrac{9}{2}-\tfrac{5}{2}=\tfrac{4}{2}=2,$$

and for the greater

$$\tfrac{9}{2}+\tfrac{5}{2}=\tfrac{14}{2}=7.$$

4. I have denoted in the above the less of the two parts by x, and I have deduced from it the greater. If it were required to find directly this last, it should be observed, that representing it by x, the other will be $x-b$, since we pass from the greater to the less by subtracting the excess of the first above the second; the number to be divided will then be expressed by $x+x-b$, or by $2x-b$, and we have consequently

$$2x-b=a.$$

This result makes it evident that $2x$ exceeds the quantity a

by the quantity b, and that consequently $2x = a + b$. By taking the half of $2x$ and of the quantity which is equal to it, we obtain for the value of x

$$x = \frac{a}{2} + \frac{b}{2},$$

which gives the same rule as the above for determining the greater of the two parts sought. I will not stop to deduce from it the expression for the smaller.

The same relation between the numbers given and the numbers required may be enunciated in many different ways. That which has led to the preceding result is deduced also from the following enunciation:

Knowing the sum a *of two numbers and their difference* b, *to find each of those numbers*; since, in other words, the number to be divided is the sum of the two numbers sought, and their difference is the excess of the greater above the less. The change in the terms of the enunciation being applied to the rules found above, we have

The less of two numbers sought is equal to half of the sum minus half of the difference.

The greater is equal to half of the sum plus half of the difference.

5. The following question is similar to the preceding, but a little more complicated.

To divide a given number into three such parts, that the excess of the mean above the least may be a given number, and the excess of the greatest above the mean may be another given number.

For the sake of distinctness I will first give determinate values to the known numbers.

I will suppose that the number to be divided is 230;
that the excess of the middle part above the least is 40; and,
that of the greatest above the middle one is 60.

Denoting the least part by x,
the middle one will be the least plus 40, or $x + 40$, and the greatest will be the middle one plus 60, or $x + 40 + 60$.

Now the three parts taken together must make the number to be divided; whence,

$$x + x + 40 + x + 40 + 60 = 230.$$

If the given numbers be united in one expression and the unknown ones in another, x is found three times in the result, and for the sake of conciseness we write

$$3x + 140 = 230.$$

But since it is necessary to add 140 to triple of x to make 230, it follows, that by taking 140 from 230 we have exactly the triple of x, or

$$3x = 230 - 140,$$

or

$$3x = 90,$$

whence it follows that

$$x = \tfrac{90}{3} = 30.$$

By adding to 30 the excess 40 of the middle part above the least, we have 70 for the middle part.

By adding to 70 the excess 60 of the greatest above the middle part, we have 130 for the greatest.

6. If the known numbers were different from those which I have used in the enunciation, we should still resolve the question by following the steps traced in the preceding article, but we should be obliged to repeat all the reasonings and all the operations, by which we have arrived at the number 30, because there is nothing to show how this number is composed of 230, 40, and 60. To render the solution independent of the particular values of numbers, and to show how the value of the unknown quantity is fixed by means of the known quantities, I will enunciate the problem thus;

To divide a given number a *into three such parts, that the excess of the middle one above the least shall be a given number* b, *and the excess of the greater above the middle one shall be a given number* c.

Designating as above by x the unknown quantity and making use of the common signs and the symbols a, b, c, which represent the known quantities in the question, the reasoning already given will be repeated.

The least part $= x$,
the middle part $= x + b$,
the greatest $= x + b + c$,
and the sum of these three makes the number to be divided; hence,

$$x + x + b + x + b + c = a.$$

This expression, which is so simple, may be still further abridged; for since it appears that x enters three times into the number to be divided and b twice, instead of $x + x + x$, I shall write $3x$, and instead of $+ b + b$, I shall write $+ 2b$, and it will become

$$3x + 2b + c = a.$$

From this last expression it is evident, that it is necessary to add to triple the number represented by x, double the number represented by b, and also the number c, in order to make the number a; it follows then, that if from the number a we take double the number b and also the number c, we shall have exactly the triple of x, or that

$$3x = a - 2b - c.$$

Now x being one third of three times x, we thence conclude that

$$x = \frac{a-2b-c}{3}.$$

It should be carefully observed, that having assigned no particular value to the numbers represented by a, b, c, the result to which we have come is equally indeterminate as to the value of x; it shews merely what operations it is necessary to perform upon these numbers, when a value is assigned to them, in order thence to deduce the value of the unknown quantity.

In short, the expression $\frac{a-2b-c}{3}$, to which x is equal, may be reduced to common language by writing, instead of the letters, the numbers which they represent, and instead of the signs, the kind of operation which they indicate; it will then become, as follows;

From the number to be divided, subtract double the excess of the middle part above the least, and also the excess of the greatest above the middle part, and take a third of the remainder.

If we apply this rule, we shall determine, by the simple operations of arithmetic, the least part. The number to be divided being for example 230, one excess 40, and the other 60, if we subtract as in the preceding article twice 40, or 80, and 60 from 230, there will remain 90, of which the third part is 30, as we have found already.

If the number to be divided were 520, one excess 50 and the other 120, we should subtract twice 50, or 100, and 120 from 520, and there would remain 300, a third of which or 100 would be the smallest part. The others are found by adding 50 to 100, which makes 150, and 120 more to this, which makes 270, so that the parts sought would be

100, 150, 270,

and their sum would be 520, as the question requires.

It is because the results in algebra are for the most part only an indication of the operations to be performed upon numbers in order to find others, that they are called in general *formulas.*

This question, although more complicated than that of article 1, may still be resolved by ordinary language, as may be seen in the following table, where against each step is placed a translation of it into algebraic characters.

PROBLEM.

To divide a number into three such parts, that the excess of the middle one above the least shall be a given number, and the excess of the greatest above the middle one shall be another given number.

SOLUTION.

By common language.	*By algebraic characters.*
	Let the number to be divided be denoted by a.
	the excess of the middle part above the least by b.
	the excess of the greatest above the middle one by c.
	The least part being x.
The middle part will be the least, plus the excess of the mean above the least.	The middle part will be $x + b$.
The greatest part will be the middle one, plus the excess of the greatest above the middle one. The three parts will together form the number proposed.	The greatest will be $x + b + c$.
Whence the least part, plus the least part, plus the excess of the middle one above the least, plus also the least part, plus the excess of the middle one above the least, plus the excess of the greatest above the middle one, will be equal to the number to be divided.	Whence $x + x + b + x + b + c = a$.

Whence three times the least part, plus twice the excess of the middle part above the least, plus also the excess of the greatest above the middle one, will be equal to the number to be divided. $\Big\}\ 3x + 2b + c = a.$

Whence three times the least part will be equal to the number to be divided, minus twice the excess of the middle part above the least, and minus also the excess of the greatest above the middle one. $\Big\}\ 3x = a - 2b - c.$

Whence in fine, the least part will be equal to a third of what remains after deducting from the number to be divided twice the excess of the middle part above the least, and also the excess of the greatest above the middle one. $\Big\}\ x = \frac{a-2b-c}{3}.$

7. The signs mentioned in article 2 are not the only ones used in algebra. New considerations will give rise to others, as we proceed. It must have been observed in article 2, that the multiplication of x by 2, and in articles 5 and 6 that of x by 3 and that of b by 2, is denoted by merely writing the figures before the letters x and b without any sign between them, and I shall express it in this manner hereafter; so that a number placed before a letter is to be considered as multiplied by the number represented by that letter. $5x$, $5a$, &c. signify five times x, five times a, &c. $\frac{3}{4}x$ or $\frac{3x}{4}$, &c. signifies $\frac{3}{4}$ of x or three times x divided by 4, &c.

In general, multiplication will be denoted by writing the factors in order one after the other without any sign between them, whenever it can be done without confusion.

Thus the expressions ax, bc, &c. are equivalent to $a \times x$, $b \times c$, &c. but we cannot omit the sign when numbers are concerned, for then 3×5, the value of which is 15, becomes 35. In this case we often substitute a point in the place of the usual sign, thus, $3 \,.\, 5$.

Equations.

8. If the solution of the problems in articles 3 and 6 be examined with attention, it will be found to consist of two parts entirely distinct from each other. In the first place, we express by means of algebraic characters the relations established by the nature of the question between the known and unknown quantities, from which we infer the equality of two quantities among themselves; for instance, in article 3 the quantities $2x + b$ and a, and in article 6 the quantities $3x + 2b + c$ and a.

We afterwards deduce from this equality a series of consequences, which terminate in showing the unknown quantity x to be equal to a number of known quantities connected together by operations, that are familiar to us; this is the second part of the solution.

These two parts are found in almost every problem which belongs to algebra. It is not easy, however, at present to give a rule adapted to the first part, which has for its object to reduce the conditions of the question to algebraic expressions. To be able to do this well, it is necessary to become familiar with the characters used in algebra, and to acquire a habit of analyzing a problem in all its circumstances, whether expressed or implied. But when we have once formed the two numbers, which the question supposes equal, there are regular steps for deducing from this expression the value of the unknown quantity, which is the object of the second part of the solution. Before treating of these I shall explain the use of some terms which occur in algebra.

An *equation* is an expression of the equality of two quantities.

The quantities which are on one side of the sign $=$ taken together are called a *member*; an equation has two *members*.

That which is on the left is called the *first member*, and the other the *second*.

In the equation $2x + b = a$, $2x + b$ is the *first member*, and a is the *second member*.

The quantities, which compose a member, when they are separated by the sign $+$ or $-$, are called *terms*.

Thus, the first member of the equation $2x + b = a$ contains two terms, namely, $2x$ and $+ b$.

The equation $\frac{3}{4}x + 7 = 8x - 12$ has two terms in each of its members, namely,

$\frac{3}{4}x$ and -7 in the first,
$8x - 12$ in the second.

Although I have taken at random, and to serve for an example merely, the equation $\frac{3}{4}x + 7 = 8x - 12$, it is to be considered, as also every other of which I shall speak hereafter, as derived from a problem, of which we can always find the enunciation by translating the proposed equation into common language. This under consideration becomes,

To find a number x *such, that by adding* 7 *to* $\frac{3}{4}$x, *the sum shall be equal to* 8 *times* x *minus* 12.

Also the equation $ax + bc - cx = ac - bx$, in which the letters a, b, c, are considered as representing known quantities, answers to the following question;

To find a number x *such, that multiplying it by a given number* a, *and adding the product of two given numbers* b *and* c, *and subtracting from this sum the product of a given number* c *by the number* x, *we shall have a result equal to the product of the numbers* a *and* c, *diminished by that of the numbers* b *and* x.

It is by exercising one's self frequently in translating questions from ordinary language into that of algebra, and from algebra into ordinary language, that one becomes acquainted with this science, the difficulty of which consists almost entirely in the perfect understanding of the signs and the manner of using them.

To deduce from an equation the value of the unknown quantity, or to obtain this unknown quantity by itself in one member and all the known quantities in the other, is called *resolving* the equation.

As the different questions, which are solved by algebra, lead to equations more or less compounded, it is usual to divide them into several kinds or *degrees*. I shall begin with *equations of the first degree*. Under this denomination are included those equations in which the unknown quantities are neither multiplied by themselves nor into each other.

Of the resolution of equations of the first degree, having but one unknown quantity.

9. We have already seen that to resolve an equation is to arrive at an expression, in which the unknown quantity alone is

one member is equal to known quantities combined together by operations which are easily performed. It follows then, that in order to bring an equation to this state, it is necessary to *free* the unknown quantity from known quantities with which it is connected. Now the unknown quantity may be united to known quantities in three ways;

1. By addition and subtraction, as in the equations,

$$x + 5 = 9 - x$$
$$a + x = b - x.$$

2. By addition, subtraction, and multiplication, as in the equations,

$$7x - 5 = 12 + 4x,$$
$$ax - b = cx + d;$$

3. Lastly, by addition, subtraction, multiplication, and division, as in the equations,

$$\frac{5x}{3} + 8 = \frac{11}{12}x + 9,$$

$$\frac{ax}{b} + cx - d = \frac{mx}{n} + \frac{p}{q}.$$

The unknown quantity is freed from additions and subtractions, where it is connected with known quantities, by collecting together into one member all the terms in which it is found; and for this purpose it is necessary to know how to transpose a term from one member to the other.

10. For example, in the equation

$$7x - 5 = 12 + 4x,$$

it is necessary to transpose $4x$ from the second member to the first, and the term -5 from the first member to the second. In order to this, it is obvious, that by cancelling $+ 4x$ in the second member, we diminish it by the quantity $4x$, and we must make the same subtraction from the first member, to preserve the equality of the two members; we write then $- 4x$ in the first member, which becomes $7x - 5 - 4x$ and we have

$$7x - 5 - 4x = 12.$$

To cancel -5 in the first member, is to omit the subtraction of 5 units, or in other words, to augment this member by 5 units; to preserve the equality then we must increase the second member by 5 units, or write $+ 5$ in this member, which will make it $12 + 5$; we have then

$$7x - 4x = 12 + 5.$$

By performing the operations indicated there will result the equation

$$8x = 17.$$

From this mode of reasoning, which may be applied to any example whatever, it is evident, that to cancel in a member a term affected with the sign +, which of course augments this member, it is necessary to subtract the term from the other member, or to write it with the sign —; that on the contrary when the term to be effaced has the sign minus, as it diminishes the member to which it belongs, it is necessary to augment the other member by the same term, or to write it with the sign +; whence we obtain this general rule;

To transpose any term whatever of an equation from one member to the other, it is necessary to efface it in the member where it is found, and to write it in the other with the contrary sign.

To put this rule in practice, we must bear in mind that the first term of each member, when it is preceded by no sign, is supposed to have the sign plus. Thus, in transposing the term cx of the literal equation $ax - b = cx + d$ from the second member to the first, we have

$$ax - b - cx = d;$$

transposing also $-b$ from the first member to the second, it becomes

$$ax - cx = d + b.$$

11. By means of this rule, we can unite together in one of the members all the terms containing the unknown quantity, and in the other all the known quantities; and under this form the member, in which the unknown quantity is found, may always be decomposed into two factors, one of which shall contain only known quantities, and the other shall be the unknown quantity by itself.

This process suggests itself immediately, whenever the proposed equation is numerical and contains no fractions, because then all the terms involving the unknown quantity may be reduced to one. If we have, for example, $10x + 7x - 2x = 25 + 7$ by performing the operations indicated in each member, we shall have in succession

$$17x - 2x = 32,$$

$$15x = 32;$$

and $15x$ is resolved into two factors 15 and x; we have then

the unknown factor x by dividing the number 32, which is equal to the product $15x$ by the given factor 15, thus,

$$x = \tfrac{32}{15}.$$

This resolution is effected in like manner in the literal equations of the form

$$ax = bc;$$

because the term ax signifies the product of a by x; we hence conclude, that

$$x = \frac{bc}{a}.$$

Let there be the equation

$$ax - bx + cx = ac - bc.$$

which contains three terms involving the unknown quantity. Since ax, bx, cx, represent the products respectively of x by the quantities a, b, and c, the expression $ax - bx + cx$ translated into ordinary language is rendered as follows,

From x *taken first, so many times as there are units in* a, *subtract so many times* x *as there are units in* b, *and add to the result the same quantity* x, *taken so many times as there are units in* c.

It follows then on the whole, that the unknown quantity x is taken so many times as there are units in the difference of the numbers a and b, augmented by the number c, that is to say, so many times as is denoted by the number $a - b + c$; the two factors of the first member are therefore $a - b + c$ and x; we have then

$$x = \frac{ac - bc}{a - b + c}.$$

From this reasoning, which may be applied to every other example, it is evident, that *after collecting together into one member the different terms containing the unknown quantity, the factor by which the unknown quantity is multiplied, is composed of all those quantities by which it is separately multiplied, arranged with their proper signs, and the unknown quantity is found by dividing all the terms of the known member by the factor which is thus obtained.*

According to this rule, the equation $ax - 3x = bc$ gives

$$x = \frac{bc}{a - 3}.$$

Also the equation $x + ax = c - d$ is reduced to

$$x = \frac{c - d}{1 + a},$$

for it is necessary to observe that the letter x, taken singly, must be regarded as multiplied by one. It is besides manifest that in $x+ax$, the unknown quantity x is contained once more than in ax, and is consequently multiplied by $1+a$.

12. It is evident that if there be a factor, which is common to all the terms of an equation, it may be dropped without destroying the equality of the two expressions, since it is merely dividing by the same number all the parts of the two quantities which are by supposition equal to each other.

Let there be, for example, the equation

$$6abx - 9bcd = 12bdx + 15abc,$$

I observe in the first place, that the numbers 6, 9, 12 and 15 are divisible by 3, and by suppressing this factor, I merely take a third part of all the quantities which compose the equation.

I have after this reduction,

$$2abx - 3bcd = 4bdx + 5abc.$$

I observe, moreover, that the letter b, combined in each term as a multiplier, is a factor common to all the terms; by cancelling it the equation becomes

$$2ax - 3cd = 4dx + 5ac.$$

Applying the rules given in articles 10 and 11, I deduce successively

$$2ax - 4dx = 5ac + 3cd,$$

$$x = \frac{5ac+3cd}{2a-4d}.$$

13. I now proceed to equations, the terms of which have divisors. These may be solved by the preceding rules whenever the unknown quantity does not enter into the denominators; but it is often more simple to reduce all the terms to the same denominator which may then be cancelled.

Let there be, for example, the equation

$$\frac{2x}{3} + 4 = \frac{4x}{5} + 12 - \frac{5x}{7}.$$

Arithmetic furnishes rules for reducing fractions to the same denominator, and for converting whole numbers into fractions of a given kind. (*Arith.* 79, 69.) Let all the terms of the proposed equation be transformed by these rules into fractions of the same denominator, beginning with the fractions, which are

$$\frac{2x}{3}, \frac{4x}{5}, \frac{5x}{7}.$$

I convert them by the first of the rules cited into the following;

$$\frac{5\times7\times2x}{3\times5\times7},\qquad\frac{3\times7\times4x}{3\times5\times7},\qquad\frac{3\times5\times5x}{3\times5\times7}.$$

Since, for converting the whole numbers 4 and 12 into fractions, nothing more is necessary than to multiply them by the common denominator of the fractions, namely, $3\times5\times7$; we have

$$3\times5\times7\times4,\qquad3\times5\times7\times12.$$

By placing all these terms in order in the proposed equation, it will become

$$\frac{5\times7\times2x}{3\times5\times7}+\frac{3\times5\times7\times4}{3\times5\times7}$$
$$=\frac{3\times7\times4x}{3\times5\times7}+\frac{3\times5\times7\times12}{3\times5\times7}-\frac{3\times5\times5x}{3\times5\times7}.$$

The denominator may now be cancelled, since by doing it we only multiply all the parts of the equation by this denominator, (*Arith.* 54), which does not destroy the equality of the members. It will become then

$$5\times7\times2x+3\times5\times7\times4$$
$$=3\times7\times4x+3\times5\times7\times12-3\times5\times5x,$$

or
$$70x+420=84x+1260-75x,$$

an equation without a denominator from which we deduce the value of x by the preceding rules.

It is evident from inspection, as also from the mere application of the arithmetical rules referred to, that in the above operation *the numerators of each fraction must be multiplied by the product of the denominators of all the others, the whole numbers by the product of all the denominators; then no account need be taken of the common denominators of the fractions thus obtained.*

The equation $70x+420=84x+1260-75x$, becomes successively

$$70x+75x-84x=1260-420,$$
$$61x=840,$$
$$x=\tfrac{840}{61}=13\tfrac{47}{61}.$$

The same process is applicable to literal equations, it being observed, that it is necessary only to indicate the multiplications, which are actually performed when numbers are concerned.

Let there be, for example, the equation

$$\frac{ax}{b}-c=\frac{dx}{e}+\frac{fg}{h};$$

we deduce from it

$$eh\times ax-beh\times c=bh\times dx+be\times fg,$$

a result which may be more simply expressed by placing the factors of each product one after the other without any sign between them, according to the method given in article 7; and by arranging the letters in alphabetical order, they are more easily read, it then becomes,

$$aehx - bceh = bdhx + befg,$$

from which is deduced

$$aehx - bdhx = befg + bceh,$$

and

$$x = \frac{befg + bceh}{aeh - bdh}.$$

14. Although no general and exact rule can be given for forming the equation of any question whatever, there is notwithstanding, a precept of extensive use, which cannot fail to lead to the proposed object. It is this;

To indicate by the aid of algebraic signs upon the known quantities represented either by numbers or letters, and upon the unknown quantities represented always by letters, the same reasonings and the same operations, which it would have been necessary to perform in order to verify the values of the unknown quantities, had they been known.

In making use of this precept, it is necessary, in the first place, to determine with care what are the operations which are contained in the enunciation of the question, either directly or by implication; but this is the very thing which constitutes the difficulty of *putting a question into an equation.*

The following examples are intended to illustrate the above precept. I have taken the two first from among the questions which are solved by arithmetic, in order to show the advantage of the algebraic method.

1. *Let there be two fountains, the first of which running for $2\frac{1}{2}$h. fills a certain vessel, and the second fills the same vessel by running $3\frac{1}{4}$h. what time will be employed by both the fountains running together in filling the vessel?*

If the time were given we should verify it by calculating the quantities of water discharged by each fountain, and adding them together we should be certain, that they would be equal to the whole content of the vessel.

To form the equation we denote the unknown time by x, and we indicate upon x the operations implied by the question; but in order to render the solution independent of the given num-

bers, and at the same time to abridge the expression where fractions are concerned, we will represent them also by letters, a being written instead of $2\frac{1}{2}$h. and b instead of $3\frac{3}{4}$h.

This being supposed, by putting the capacity of the vessel equal to unity, it is evident, that,

The first fountain, which will fill it in a number of hours denoted by a, will discharge into it in one hour a quantity of water expressed by the fraction $\frac{1}{a}$, and that consequently, in a number x of hours, it will furnish the quantity $x \times \frac{1}{a}$, or $\frac{x}{a}$. (*Arith.* 53).

The second fountain, which will fill the same vessel in a number of hours described by b, will discharge into it in one hour a quantity of water expressed by the fraction $\frac{1}{b}$, and consequently in a number x of hours, it will furnish the quantity $x \times \frac{1}{b}$, or $\frac{x}{b}$.

The whole quantity of water then furnished by the two fountains, will be

$$\frac{x}{a} + \frac{x}{b};$$

and this quantity must be equal to the content of the vessel, which was considered as unity; we have then the equation,

$$\frac{x}{a} + \frac{x}{b} = 1.$$

This equation reduced by the foregoing rules, becomes

$$bx + ax = ab,$$

$$x = \frac{ab}{b+a}.$$

The last formula gives this simple rule for resolving every case of the proposed question.

Divide the product of the numbers, which denote the times employed by each fountain in filling the vessel, by the sum of these numbers; the quotient expresses the time required by both the fountains running together to fill the vessel.

Applying this rule to the particular case under consideration, we have

$$2\tfrac{1}{2} \times 3\tfrac{3}{4} = \tfrac{5}{2} \times \tfrac{15}{4} = \tfrac{75}{8},$$

$$2\tfrac{1}{2} + 3\tfrac{3}{4} = \tfrac{5}{2} + \tfrac{15}{4} = \tfrac{20}{8} + \tfrac{30}{8} = \tfrac{50}{8},$$

whence $$x = \tfrac{78}{52} = \tfrac{3}{2}.$$

2. *Let a be a number to be divided into three parts, having among themselves the same ratios as the given numbers* m, n, *and* p.

It is evident that the verification of the question would be as follows;

denoting the 1st part by x, we have

$$m : n :: x : \text{the 2d part} = \frac{nx}{m}, \ (\textit{Arith. } 116.)$$

$$m : p :: x : \text{the 3d part} = \frac{px}{m};$$

the three parts added together must make the number to be divided. We have then the equation

$$x + \frac{nx}{m} + \frac{px}{m} = a.$$

By reducing all the terms to the denominator m, it becomes

$$mx + nx + px = am,$$

and we deduce from this

$$x = \frac{am}{m+n+p}.$$

This result is nothing more nor less than an algebraic expression of *the rule of Fellowship*, (*Arith.* 124); for by regarding the numbers m, n, p, as denoting the stocks of several persons trading in company, $m+n+p$ is the whole stock, a the gain to be divided, and the equation

$$x = \frac{ma}{m+n+p},$$

shows that *a share is obtained by multiplying the corresponding stock into the whole gain, and dividing the product by the sum of the stocks;* which reduced to a proportion, becomes

the whole stock : a particular stock

:: the whole gain : to the particular gain.

15. To form an equation from the following question, requires an attention to some things, which have not yet been considered.

A fisherman, to encourage his son, promises him 5 cents for each throw of the net in which he shall take any fish, but the son, on the other hand, is to remit to the father 3 cents for each unsuccessful throw. After 12 throws the father and the son settle their account, and the former is found to owe the latter 28 cents. What was the number of successful throws of the net?

If we represent this number by x, the number of unsuccessful ones will be $12-x$; and if these numbers were given, we should verify them by multiplying 5 cents by the first, to obtain what the father was bound to pay the son, and 3 cents by the second, to find what the son engaged to return to the father. The first number ought to exceed the second by 28 cents, which the father owed the son.

We have for the first number x times 5 cents, or $5x$. With respect to the second, there is some difficulty. How are we to obtain the product of 3 by $12-x$? If instead of x we had a given number, we should first perform the subtraction indicated, and then multiply 3 by the remainder; but this cannot be done at present, and we must endeavour to perform the multiplication before the subtraction, or at least, to give the expression an entire algebraic form, similar to that of equations that are readily solved.

With a little attention we shall see, that by taking 12 times the number three, we repeat the number 3 so many times too much, as there are units in the number x, by which we ought first to have diminished the multiplier 12, so that the true product will be 36 diminished by 3 taken x times or $3x$,

or more simply $36-3x$.

This conclusion may be verified by giving to x a numerical value. If for example x were equal to 8, we should have 3 to be taken 12 times -8 times, and if we neglect -8 times, we should make the result 8 times the number 3 too much; the true product then will be

$$3\times 12-3\times 8=36-24=12.$$

This result agrees with that which would arise from first subtracting 8 from 12; for then

$$12-8=4, \quad \text{and} \quad 3\times 4=12.$$

This being admitted, since the money due from the father to the son is expressed by $5x$, and that which the son owes the father by $36-3x$, the second number must be subtracted from the first in order to obtain the remainder 28; but here is another difficulty; how shall we subtract $36-3x$ from $5x$, without having first subtracted $3x$ from 36?

We shall avoid this difficulty by observing, that if we neglect the term $-3x$, and subtract from $5x$ the entire number 36, we shall have taken *necessarily* $3x$ too much, since it is only what

remains after having diminished 36 by $3x$ ought to be subtracted from $5x$; so that the difference $5x - 36$ ought to be augmented by $3x$ in order to form the quantity that should remain after having taken from $5x$ the number denoted by $36 - 3x$. This quantity will then be

$$5x - 36 + 3x;$$

and we have the equation

$$5x - 36 + 3x = 28,$$

which becomes successively

$$8x - 36 = 28,$$
$$8x = 28 + 36,$$
$$8x = 64,$$
$$x = \tfrac{64}{8} = 8.$$

There have been then 8 successful throws of the net and 4 unsuccessful ones.

Indeed 8 throws at 5 cents a throw give 40 cents,
4 throws at 3 cents a throw give 12

difference 28

as required by the conditions of the question.

To render the solution general, let a represent the sum given by the father to the son for each successful throw of the net, and b the sum returned by the son for each unsuccessful one, and c the total number of throws, and d the sum received on the whole by the son. If x be put equal to the number of successful throws, $c - x$ will express the number of unsuccessful ones; each throw of the former kind being worth to the son a sum a, x throws would be worth $a \times x$ or ax, and the unsuccessful throws would be worth to the father the sum b multiplied by the number $c - x$.

The reasoning by which we have found the parts of the product of 3 by $12 - x$, applies equally to the general case. If we neglect in the first place $-x$ in forming the product bc of b by the whole of c, the sum b will be repeated x times too much, and consequently the true product will be $bc - bx$.

In order to subtract this product from the sum ax, it is necessary to observe, as in the numerical example, that if we subtract the whole of the quantity bc we take the quantity bx too much, by which the former ought to have been first diminished, and that consequently the true remainder is not merely $ax - bc$, but $ax - bc + bx$.

As this sum is equal to d, we have the equation

$$ax - bc + bx = d,$$

which gives

$$ax + bx = d + bc,$$

$$x = \frac{d+bc}{a+b}.$$

As this general formula indicates what operations are to be performed upon the numbers a, b, c, d, in order to obtain the unknown quantity x, we may reduce it to a rule or carefully write instead of the letters a, b, c, d, the numbers given. This last process is called *substituting* the values of the given quantities, or *putting the formula into numbers*. Applying here those of the foregoing example, we have

$$x = \frac{28 + 3 \times 12}{5+3};$$

by performing the operations indicated, it becomes

$$x = \frac{28+36}{8} = \frac{64}{8} = 8.$$

Methods for performing, as far as is possible, the operations indicated upon quantities that are represented by letters.

16. From the preceding question it is evident, that in certain cases a multiplication indicated upon the sum or difference of several quantities cannot be separated into parts; and in art. 11, we have exactly the reverse, by resolving the quantity $ax - bx + cx$, which represents the result of several multiplications, followed by additions and subtractions, into the two factors $a - b + c$ and x, which indicate only a single multiplication preceded by addition and subtraction. The reasoning pursued in these two circumstances, will suggest rules for performing, upon quantities represented by letters, operations which are called *algebraic multiplication* and *division*, from the analogy which they have with the corresponding operations of arithmetic.

We have also by the same analogy two algebraic operations, which bear the names of *addition* and *subtraction*, in which the object is to unite several algebraic expressions in one, or to take one expression from another. But these operations, like the preceding, differ from those of arithmetic in this, that their results are, for the most part, only indications of the operations to be performed; they present only a transformation of the

operations originally indicated into others, which produce the same effect. All that is done, is either to simplify the expressions, or to give them a proper form for exhibiting the conditions that are to be fulfilled.

In order to explain these operations, we give the name of *simple quantities* to those which consist only of one term, as $+2a$, $-3ab$, &c. *binomials* to those which consist of two, as $a+b$, $a-b$, $5a-2x$, &c. *trinomials* to those which consist of three terms, *quadrinomials* to those which consist of four terms, and *polynomials* to those which consist of more than four terms. It may be observed also, that we call polynomials *compound quantities*.

Of the addition of algebraic quantities.

17. The addition of simple quantities is performed by writing them one after the other, with the sign $+$ between them; thus, a added to b is expressed by $a+b$. But when it is proposed to add together several algebraic expressions, we aim at the same time to simplify the result by reducing it to as small a number of terms as possible by uniting several of the terms in one. This is done in articles 2 and 5, by reducing the quantity $x+x$ to $2x$, and the quantity $x+x+x$ to $3x$. It can take place only with respect to quantities expressed by the same letters, and which are for this reason called *similar* quantities. A literal quantity that is repeated any number of times is regarded as a unit, it is thus, that the quantities $2a$ and $3a$ considered as two and three units of a particular kind, form when added $5a$, or 5 units of the same kind. Also $4ab$ and $5ab$ make $9ab$.

In this case, the addition is performed with respect to the figures which precede the literal quantity, and which show how many times it is repeated. These figures are called *coefficients*. The coefficient then is the multiplier of the quantity before which it is placed, and it must be recollected, that when there is none expressed, unity is understood; for $1a$ is the same as a.

18. When it is proposed to unite any quantities whatever, as

$$4a+5b \text{ and } 2c+3d,$$

the sum total ought evidently to be composed of all the parts joined together; we must write then

$$4a+5b+2c+3d.$$

If we have on the contrary

$$4a+5b \quad \text{and} \quad 2c-3d.$$

The sign $-$ must be retained in the sum, to mark as subtractive the quantity $3d$, which, as it is to be taken from $2c$, must necessarily diminish by so much the sum formed by uniting $2c$ with the first of the quantities proposed; we have then,

$$4a+5b+2c-3d.$$

From these two examples it is evident, that *in algebra the addition of polynomials is performed by writing in order, one after the other, the quantities to be added with their proper signs, it being observed that the terms which have no signs before them are considered as having the sign* $+$.

The above operation is, properly speaking, only an indication by which the union of two compound quantities is made to consist in the addition and subtraction of a certain number of simple quantities; but if the quantities to be added contained similar terms, these terms might be united by performing the operation upon their coefficients.

Let there be, for example, the quantities

$$4a+9b-2c,$$
$$2a-3c+4d,$$
$$7b+c-e;$$

the sum indicated would be, according to the rule just given,

$$4a+9b-2c+2a-3c+4d+7b+c-e.$$

But the terms $4a$, $+2a$, being formed of similar quantities, may be united in one sum equal to $6a$.

Also the terms $+9b$, $+7b$ give $+16b$.

The terms $-2c$ and $-3c$, being both subtractive, produce on the whole, the same effect as the subtraction of a quantity equal to their sum, that is to say, as the subtraction of $5c$; and as by virtue of the term $+c$, we have another part c to be added, there will remain therefore to be subtracted only $4c$.

The sum of the expressions proposed then, will be reduced to

$$6a+16b-4c+4d-e.$$

The last operation exhibited above, by which all similar terms are united in one, whatever signs they have, is called *reduction. It is performed by taking the sum of similar quantities having the sign* $+$, *that of similar quantities having the sign* $-$, *and subtracting the less of the two sums from the greater, and giving to the remainder the sign of the greater.*

It is to be remarked, that reduction is applicable to all algebraic operations.

The following examples of addition, with their answers, are intended as an exercise for the learner.

1. To add the quantities

$$\begin{array}{l} 7m+3n-14p+17r \\ 3a+9n-11m+2r \\ 5p-4m+8n \\ 11n-2b-m-r+s \end{array}$$

Answer, $7m+3n-14p+17r+3a+9n-11m+2r+5p-4m+8n+11n-2b-m-r+s.$

By making the reduction, this quantity becomes

$$-9m+31n-9p+18r+3a-2b+s,$$

or $$31n-9m-9p+18r+3a-2b+s,$$

by beginning with a term having the sign $+$.

2. To add the quantities

$$\begin{array}{l} 11bc+4ad-8ac+5cd \\ 8ac+7bc-2ad+4mn \\ 2cd-3ab+5ac+an \\ 9an-2bc-2ad+5cd \end{array}$$

$$11bc+4ad-8ac+5cd+8ac+7bc-2ad+4mn$$
$$2cd-3ab+5ac+an+9an-2bc-2ad+5cd.$$

By reducing this quantity it becomes

$$16bc+5ac+12cd+4mn-3ab+10an.$$

Of the Subtraction of Algebraic Quantities.

20. The subtraction of single quantities, according to established usage, is represented by placing the sign $-$ between the quantity to be subtracted, and that from which it is to be taken; b subtracted from a is written $a-b$.

When the quantities are similar, the subtraction is performed directly by means of the coefficients.

If $3a$ be subtracted from $5a$, we have for a remainder $2a$.

With regard to the subtraction of polynomials, it is necessary to distinguish two cases.

1. If the terms of the quantity to be subtracted have each the sign $+$, we must clearly give to each the sign $-$, since it is required to deduct successively all the parts of the quantity to be subtracted.

If for example, from $5a - 9b + 2c$ we would take $2d + 3e + 4f$, we must write

$$5a - 9b + 2c - 2d - 3e - 4f.$$

2. If any of the terms of the quantity to be subtracted have the sign —, we must give them the sign plus. Indeed, if from the quantity a we would take $b - c$ and should first write $a - b$, we should thus diminish a by the whole quantity b; but the subtraction ought to have been performed after having first diminished b by the quantity c; we have taken therefore this last quantity too much, and it is necessary to restore it with the sign $+$, which gives for the true result $a - b + c$.

This reasoning, which may be applied to all similar cases, shows that the sign — of c must be changed into the sign $+$; and by connecting this result with the preceding, we conclude, that *the subtraction of algebraic quantities is performed by writing them in order after the quantities, from which they are to be taken, having first changed the signs + into — and the signs — into +.*

After this rule has been applied, the quantities are to be reduced when they will admit of it, according to the precept given in article 19, as may be seen in the following examples;

1. To subtract from $17a + 2m - 9b - 4c + 23d$
the quantity $51a - 27b + 11c - 4d$.

Result, $17a + 2m - 9b - 4c + 23d$
$- 51a + 27b - 11c + 4d$.

When reduced it becomes

$$-34a + 2m + 18b - 15c + 27d,$$

or rather $2m - 34a + 18b - 15c + 27d$.

2. To subtract from $5ac - 8ab + 9bc - 4am$
the quantity $8am - 2ab + 11ac - 7cd$.

Result $5ac - 8ab + 9bc - 4am$
$- 8am + 2ab - 11ac + 7cd$.

Reduced it becomes

$$-6ac - 6ab + 9bc - 12am + 7cd$$

or $9bc - 6ac - 6ab - 12am + 7cd$.

Of the multiplication of algebraic quantities.

21. So far as letters are considered as expressing the numerical values of the quantities for which they stand, multiplication in algebra is to be regarded like multiplication in arithmetic.

(*Arith.* 21, 66.) Thus, *to multiply* a *by* b *is to compound with the quantity represented by* a *another quantity, in the same manner as the quantity represented by* b *is with unity.*

We have already explained, in articles 2 and 7, the signs used to indicate multiplication; and the product of a by b is expressed by $a \times b$, or by $a \,.\, b$, or lastly, by $a\,b$.

We have often occasion to express several successive multiplications, as that of a by b, and that of the product $a\,b$ by c, also that of this last product by d, and so on. In this case, it is evident, that the last result is a number having for *factors* the numbers a, b, c, d, (*Arith.* 22); and to give a general expression of this method, *we indicate the product by writing the factors composing it in order, one after the other, without any sign between them*; we have accordingly the expression $a\,b\,c\,d$.

Reciprocally every expression, such as $a\,b\,c\,d$ formed of several letters written in order one after the other, designates always the product of the numbers represented by these letters.

I have already availed myself of this method, in which the numerical coefficients are also included, since they are evidently factors of the quantity proposed. Indeed $15\,a\,b\,c\,d$, designating the quantity $a\,b\,c\,d$ taken 15 times, expresses likewise the product of the five factors 15, a, b, c, d.

It follows from this, that in order to indicate the multiplication of several simple quantities, such as $4\,a\,b\,c$, $5\,d\,e\,f$, $3\,m\,n$, it is necessary to write the quantities in order, one after the other, without any sign between them, and it becomes

$$4\,a\,b\,c\,5\,d\,e\,f\,3\,m\,n\,;$$

but since, as is shown in arithmetic, (art. 82) the order of the factors of a product may be changed at pleasure without altering the value of this product, we may avail ourselves of this principle, to bring together the numerical factors, the multiplication of which is performed by the rules of arithmetic; to express then this product, as indicated in the order $4\,.\,5\,.\,3\;a\,b\,c\,d\,e\,f\,m\,n$, we multiply together the numbers 4, 5, 3, which give simply

$$60\,a\,b\,c\,d\,e\,f\,m\,n.\ *$$

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* As the use of algebraic symbols abridges very much the demonstration of this proposition. I have thought it proper to suggest here a method by these symbols.

If we write the product $a\,b\,c\,d\,e\,f$ as follows, $a\,b\,c \times d\,e \times f$, and

23. The expression of the product may be much abridged when it contains equal factors. Instead of writing several times in order, the letter which represents one of the factors, it need be written only once with a number annexed, showing how many times it ought to have been written as a factor; but as this number indicates successive multiplications, it ought to be carefully distinguished from a coefficient, which indicates only additions. For this reason, it is placed on the right of the letter and a little higher up, while a coefficient is always placed on the left and on the same line.

Agreeably to this method, the product of a by a, which would be indicated according to article 21, by $a\,a$ becomes a^2. The 2 raised, denotes that the number, designated by the letter a, is twice a factor in the expression, to which it belongs. It ought not to be confounded with $2a$ which is only an abbreviation of $a+a$. To render evident the error, which would arise from mistaking one for the other, it is sufficient to substitute numbers instead of the letters. If we have, for example, $a=5$, $2a$ would become $2\,.\,5=10$, and $a^2=a\times a=5\,.\,5=25$.

Extending this method we should denote a product in which a is three times a factor by writing a^3 instead of $a\,a\,a$; also a^5 represents a product in which a is five times a factor, and is equivalent to $a\,a\,a\,a\,a$.

24. The products formed in this manner by the successive multiplications of a quantity, are called in general *powers* of that quantity.

The quantity itself, as a, is called the first power.

The quantity multiplied by itself, as aa, or a^2, is the second power. It is called also the *square*.

The quantity multiplied by itself twice in succession, as aaa, or a^3, is the third power, and is called also the *cube*.*

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change the order of the factors of the product to $d\,e$ instead of $e\,d$, (*Arith.* 27.) it becomes $a\,b\,c\times e\,d\times f$, or $a\,b\,c\,d\,e\,f$. It is evident that we may, by analyzing the product differently, produce any change which we wish in the order of the factors of the product in question.

* The denominations *square* and *cube* refer to geometrical considerations. They interrupt the uniformity in the nomenclature of products formed by equal factors, and are very improper in algebra. But they are frequently used for the sake of conciseness.

In general, any power whatever is designated by the number of equal factors from which it is formed; a^5 or $a\,a\,a\,a\,a$ is the *fifth* power of a.

I take the number 3 to illustrate these denominations, and I have

1st. power	3
2d.	$3 \,.\, 3 = 9$
3d.	$3 \,.\, 3 \,.\, 3 = 9 \,.\, 3 = 27$
4th.	$3 \,.\, 3 \,.\, 3 \,.\, 3 = 27 \,.\, 3 = 81$
5th.	$3 \,.\, 3 \,.\, 3 \,.\, 3 \,.\, 3 = 81 \,.\, 3 = 243$
&c.	

The number which denotes the power of any quantity is called the *exponent* of this quantity.

When the exponent is equal to unity it is not written; thus a is the same as a^1.

It is evident then, that *to find the power of any number, it is necessary to multiply this number by itself as many times less one, as there are units in the exponent of the power.*

25. As the exponent denotes the number of equal factors which form the expression of which it is a part, and as the product of two quantities must have each of these quantities as factors; it follows that the expression a^5 in which a is five times a factor, multiplied by a^3, in which a is three times a factor, ought to give a product in which a is eight times a factor, and consequently expressed by a^8, and that in general *the product of two powers of the same number ought to have for an exponent the sum of those of the multiplicand and multiplier.*

26. It follows from this, *that when two simple quantities have common letters, we may abridge the expression of the product of these quantities by adding together the exponents of such letters of the multiplicand and multiplier.*

For example, the expression of the product of the quantities $a^2\, b^3\, c$ and $a^4\, b^5\, c^2\, d$, which would be $a^2\, b^3\, c\, a^4\, b^5\, c^2\, d$, by the foregoing rule, art. 21, is abridged by collecting together the factors designated by the same letter, and

$$a^2\, a^4\, b^3\, b^5\, c\, c^2\, d,$$

becomes $\quad a^6\, b^8\, c^3\, d,$

by writing $\quad a^6$ instead of $a^2\, a^4$

b^8 instead of $b^3\, b^5$

c^3 instead of $c\, c^2$ or of $c^1\, c^2$.

27. As we distinguish powers by the number of equal factors from which they are formed, so also we denote any products by the number of simple factors or *firsts* which produce them; and I shall give to these expressions the name of *degrees*. The product $a^2 b^3 c$, for example, will be of the sixth degree, because it contains six simple factors, viz; 2 factors a, 3 factors b, and 1 factor c. It is evident that the factors a, b, and c, here regarded as firsts, are not so, except with respect to algebra, which does not permit us to decompose them; they may, notwithstanding, represent compound numbers, but we here speak of them only with respect to their general import.*

As the coefficients expressed in letters are considered only in estimating the degree of algebraic quantities, we have regard only to the letters.

It is evident, (21, 25) that when we multiply two simple quantities the one by the other, the number which marks the degree of the product is the sum of those which mark the degree of each of the simple quantities.

28. The multiplication of compound quantities consists in that of simple quantities, each term of the multiplicand and multiplier being considered by itself; as in arithmetic we perform the operation upon each figure of the numbers which we propose to multiply. (*Arith.* 33.) The particular products added together make up the whole product. But algebra presents a circumstance which is not found in numbers. These have no negative terms or parts to be subtracted, the units, tens, hundreds, &c. of which they consist, are always considered as added together, and it is very evident, that the whole product must be composed of the sum of the products of each part of the multiplicand by each part of the multiplier.

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* We apply the term *dimensions*, generally to what I have here called *degrees*, in conformity to the analogy already pointed out in the note to page 29. This example sufficiently proves the absurdity of the ancient nomenclature, borrowed from the circumstance, that the products of 2 and 3 factors, measure respectively the areas of the surfaces and the bulks of bodies, the former of which have two and the latter three dimensions; but beyond this limit the correspondence between the algebraic expressions and geometrical figures fails, as extension can have only three dimensions.

The same is true of literal expressions when all the terms are connected together by the sign +.

The product of $a+b$
multiplied by c

is $ac+bc$,

and is obtained by multiplying each part of the multiplicand by the multiplier, and adding together the two particular products ac and bc. The operation is the same when the multiplicand contains more than two parts.

If the multiplier is composed of several terms, it is manifest that the product is made up of the sum of the products of the multiplicand by each term of the multiplier.

The product of $a+b$
multiplied by $c+d$

is $\left\{\begin{array}{l} ac+bc \\ +ad+bd; \end{array}\right.$

for by multiplying first $a+b$ by c, we obtain $ac+bc$, then by multiplying $a+b$ by the second term d of the multiplier, we have $ad+bd$, and the sum of the two results gives $ac+bc+ad+bd$ for the whole.

29. When the multiplicand contains parts to be subtracted, the products of these parts by the multiplier must be taken from the others, or in other words, have the sign — prefixed to them. For example,

the product of $a-b$
multiplied by c

is $ac-bc$;

for each time that we take the entire quantity a, which was to have been diminished by b before the multiplication, we take the quantity b too much; the product ac therefore, in which the whole of a is taken as many times as is denoted by the number c, exceeds the product sought by the quantity b, taken as many times as is denoted by the number c, that is by the product bc; we ought then to subtract bc from ac, which gives, as above,

$$ac-bc.$$

The same reasoning will apply to each of the parts of the multiplicand, that are to be subtracted, whatever may be the number, and whatever may be that of the terms of the multiplier, pro-

vided they all have the sign +. Recollecting that the terms which have no sign are considered as having the sign +, we see by the examples, that the terms of the multiplicand affected by the sign + give a product affected by the sign +, while those which have the sign — give one having the sign —. It follows from this, that *when the multiplier has the sign +, the product has the same sign as the corresponding part of the multiplicand.*

30. The contrary takes place when the multiplier contains parts to be subtracted; the products arising from these parts must be put down with a sign, contrary to that which they would have had by the above rule. This may be shown by the following example.

Let the multiplicand be $\quad a - b$
and the multiplier $\quad c - d$

the product will be $\quad \left\{ \begin{array}{l} ac - bc \\ -ad + bd; \end{array} \right.$

for the product of the multiplicand, by the first term of the multiplier, will be by the last example $ac - bc$; but by taking the whole of c for the multiplier instead of c diminished by d, we take the quantity $a - b$ so many times too much as is denoted by the number d; so that the product $ac - bc$ exceeds that sought by the product of $a - b$ by d. Now this last is, by what has been said, $ad - bd$, and in order to subtract it from the first it is necessary to change the signs (20). We have then

$$ac - bc - ad + bd \text{ for the result required.}$$

31. Agreeably to the above examples, we conclude, that *the multiplication of polynomials is performed by multiplying successively, according to the rules given for simple quantities* (21—26), *all the terms of the multiplicand by each term of the multiplier, and by observing that each particular product must have the same sign, as the corresponding part of the multiplicand, when the multiplier has the sign +, and the contrary sign when the individual multiplier has the sign —.*

If we develop the different cases of this last rule, we shall find,

1. That a term having the sign +, multiplied by a term having the sign +, gives a product having the sign +;

2. That a term having the sign —, multiplied by a term having the sign +, gives a product which has the sign —;

3. That a term having the sign +, multiplied by a term having the *sign —, gives a product* which has the sign —;

4. That a term having the sign —, multiplied by a term having the sign —, gives a product which has the sign +.

It is evident from this table, that *when the multiplicand and multiplier have the same sign, the product has the sign +, and that when they have different signs, the product has the sign —.*

To facilitate the practice of the multiplication of polynomials, I have subjoined a recapitulation of the rules to be observed.

1. *To determine the sign of each particular product according to the rule just given*; this is the rule for the signs.

2. *To form the coefficients by taking the product of those of the multiplicand and multiplier* (22); this is the rule for the coefficients.

3. *To write in order, one after the other, the different letters contained in each multiplicand and multiplier* (21); this is the rule for the letters.

4. *To give to the letters, common to the multiplicand and multiplier, an exponent equal to the sum of the exponents of these letters in the multiplicand and multiplier* (25); this is the rule for the exponents.

32. The example below will illustrate all these rules.

Multiplicand	$5a^4 - 2a^3b + 4a^2b^2$
Multiplier	$a^3 - 4a^2b + 2b^3$

$$\text{Several products.}\left\{\begin{array}{l} 5a^7 - 2a^6b + 4a^5b^2 \\ -20a^6b + 8a^5b^2 - 16a^4b^3 \\ +10a^4b^3 - 4a^3b^4 + 8a^2b^5. \end{array}\right.$$

Result reduced $\quad 5a^7 - 22a^6b + 12a^5b^2 - 6a^4b^3 - 4a^3b^4 + 8a^2b^5$.

The first line of the several products contains those of all the terms of the multiplicand by the first term a^3 of the multiplier; this term being considered as having the sign +, the products which it gives have the same signs as the corresponding terms of the multiplicand (31).

The first term $5a^4$ of the multiplicand having the sign plus, we do not write that of the first term of the product, which would be +; the coefficient 5 of a^4 being multiplied by the coefficient 1 of a^3, gives 5 for the coefficient of this product; the sum of the two exponents of the letter a is $4+3$, or 7, the first term of the product then is $5a^7$.

The second term $-2a^3b$ of the multiplicand having the sign —, the product has the sign minus; the coefficient 2 of a^3b mul-

tiplied by the coefficient 1 of a^3, gives 2 for the coefficient of the product; the exponent of the letter a, common to the two terms which we multiply, is $3 + 3$, or 6, and we write after it the letter b, which is found only in the multiplicand. The second term of the product then is $-2a^6b$.

The third term $+4a^2b^2$ gives a product affected with the sign $+$, and by the rules applied to the two preceding terms, we find it to be $+4a^5b^2$.

The second line contains the products of all the terms of the multiplicand by the second term $-4a^3b$ of the multiplier. This last having the sign $-$, all the products which it gives must have the signs contrary to those of the corresponding terms of the multiplicand; the coefficients, the letters, and the exponents are determined as in the preceding line.

The third line contains the products of all the terms of the multiplicand by the third term $+2b^2$ of the multiplier. This term having the sign $+$, all the products which it gives have the same sign as the corresponding terms of the multiplicand.

After having formed all the several products which compose the whole product, we examine carefully this last, to see whether it does not contain similar terms; if it does, we reduce them according to the rule (19), observing that two terms are similar, which consist of the same letters under the same exponents. In this example there are three reductions, viz;

$$-2a^6b \text{ and } -20a^6b, \text{ which give } -22a^6b;$$
$$+4a^5b^2 \text{ and } +8a^5b^2, \text{ which give } +12a^5b^2;$$
$$-16a^4b^3 \text{ and } +10a^4b^3, \text{ which give } -6a^4b^3.$$

These reductions being made, we have for the result the last line of the example.

See another example to exercise the learner, which is easily performed after what has been said.

$$\begin{array}{ll}
\textbf{Multiplicand} & 5a^4b^2+7a^3b^3-15a^5c+23b^2d^4-17bc^3d^2-9abcdm^2 \\
\textbf{Multiplier} & 11b^3-8c^3+5abc-2bdm \\
\hline
\begin{array}{l}\textbf{Several}\\ \textbf{products.}\end{array} & \left\{\begin{array}{l}
55a^4b^5+77a^3b^6-165a^5b^3c+253b^5d^4-187b^4c^3d^2-99ab^4cdm^2 \\
-40a^4b^2c^3-56a^3b^3c^3+120a^5c^4-184b^2c^3d^4+136bc^6d^2+72abc^4dm^2 \\
+25a^5b^3c+35a^4b^4c-75a^6bc^2+115ab^3cd^4-85ab^2c^4d^2-45a^2b^2c^2dm^2 \\
-10a^4b^3dm-14a^3b^4dm+30a^5bcdm-46b^3d^5m+34b^2c^3d^3m+18ab^2cd^2m^3
\end{array}\right. \\
\hline
\begin{array}{l}\textbf{Result}\\ \textbf{reduced.}\end{array} & \left\{\begin{array}{l}
55a^4b^5+77a^3b^6-140a^5b^3c+253b^5d^4-187b^4c^3d^2-99ab^4cdm^2-40a^4b^2c^3-56a^3b^3c^3 \\
+120a^5c^4-184b^2c^3d^4+136bc^6d^2+72abc^4dm^2+35a^4b^4c-75a^6bc^2+115ab^3cd^4-85ab^2c^4d^2 \\
-45a^2b^2c^2dm^2-10a^4b^3dm-14a^3b^4dm+30a^5bcdm-46b^3d^5m+34b^2c^3d^3m+18ab^2cd^2m^3.
\end{array}\right.
\end{array}$$

33. From the manner of proceeding in multiplication, it is evident, that if all the terms of the multiplicand are of the same degree (27), and those of the multiplier are also of the same degree, all the terms of the product will be of a degree denoted by the sum of the numbers, which mark the degree of the terms of each of the factors.

In the first example, the multiplicand is of the fourth degree, the multiplier of the third; and the product is of the seventh.

In the second example, the multiplicand is of the sixth degree, the multiplier of the third; and the product is of the ninth.

Expressions of the kind just referred to, the terms of which are all of the same degree, are called *homogeneous* expressions. The above remark, with respect to their products, may serve to prevent occasional errors, which one may commit by forgetting some of the factors in the several parts of the multiplication.

34. Algebraic operations performed upon literal quantities, as they permit us to see how the several parts of the quantities concur to form the results, often make known some general properties of numbers independent of every system of notation. The multiplications that follow, lead to conclusions of the greatest importance, and of frequent use in the subsequent parts of this work.

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ -ab-b^2 \\ \hline a^2-b^2 \end{array} \qquad\qquad \begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

$$\begin{array}{r} a^2+2ab+b^2 \\ a+b \\ \hline a^3+2a^2b+ab^2 \\ +a^2b+2ab^2+b^3 \\ \hline a^3+3a^2b+3ab^2+b^3. \end{array}$$

It appears from the first of these products, that the quantity $a+b$, multiplied by $a-b$, gives a^2-b^2; whence it is evident that, *if we multiply the sum of two numbers by their difference, the product will be the difference of the squares of these numbers.*

If we take, for example, the sum 11 of the numbers 7 and 4,

and multiply it by the difference 3 of these numbers, the product 3×11, or 33, will be equal to the difference between 49, the square of 7, and 16, the square of 4.

By the second example, in which $a + b$ is twice a factor, we learn; *that the second power, or the square of a quantity composed of two parts* a *and* b *contains the square of the first part, plus double the product of the first part by the second, plus the square of the second.*

The third example, in which we have multiplied the second power of $a + b$ by the first, shows; that, *the third power or cube of a quantity composed of two parts contains the cube of the first, plus three times the square of the first multiplied by the second, plus three times the first multiplied by the square of the second, plus the cube of the second.*

35. As we have often occasion to decompose a quantity into its factors, and as the algebraic operations are dispensed with, when it can be done, in order to exhibit the formation of the quantities to be considered, as distinctly as possible, it is necessary to fix upon some signs proper to indicate multiplication between complex quantities.

We use indeed the marks of a parenthesis to comprehend the factors of a product. The expression

$$(5a^4 - 3a^2b^2 + b^4)(4ab^2 - ac^2 + d^3)(b^2 - c^2),$$

for example, indicates the product of the compound quantities

$$5a^4 - 3a^2b^2 + b^4, \quad 4ab^2 - ac^2 + d^3 \text{ and } b^2 - c^2.$$

Bars were used formerly by some authors placed over the factors thus,

$$\overline{5a^4 - 3a^2b^2 + b^4} \times \overline{4ab^2 - ac^2 + d^3} \times \overline{b^2 - c^2};$$

but as these may happen to be too long or too short, they are liable to more uncertainty than the marks of a parenthesis, which can never admit of any doubt with respect to the quantity belonging to each factor. They have accordingly been preferred.

Of the division of algebraic quantities.

36. Algebraic division, like division in arithmetic, is to be regarded as an operation disigned to discover one of the factors of a given product, when the other is known. According to this definition, the quotient multiplied by the divisor must produce anew the dividend.

By applying what is here said to simple quantities we shall see by art. 21, that the dividend is formed from the factors of the divisor and those of the quotient; whence, *by suppressing in the dividend all the factors which compose the divisor, the result will be the quotient sought.*

Let there be, for example, the simple quantity $72\,a^5\,b^3\,c^2\,d$ to be divided by the simple quantity $9\,a^3\,b\,c^2$; according to the rule above given, we must suppress in the first of these quantities the factors of the second, which are respectively

$$9,\ a^3,\ b,\ \text{and}\ c^2.$$

It is necessary then, in order that the division may be performed, that these factors should be in the dividend. Taking them in order, we see in the first place that the coefficient 9 of the divisor, ought to be a factor of the coefficient 72 of the dividend, or that 9 ought to divide 72 without a remainder. This is in fact the case, since $72 = 9 \times 8$. By suppressing then the factor 9, there will remain the factor 8 for the coefficient of the quotient.

It follows moreover, from the rules of multiplication (25), that the exponent 5 of the letter a in the dividend, is the sum of the exponents belonging to the divisors and quotient; this last exponent therefore will be the difference between the two others, or $5 - 3 = 2$. Thus the letter a has in the quotient the exponent 2. For the same reason, the letter b has in the quotient an exponent equal to $3 - 1$, or 2. The factor c^2 being common to the dividend and divisor is to be suppressed, and we have

$$8\,a^2\,b^2\,d$$

for the quotient required.

The same will apply to every other case; we conclude then, that, *in order to effect the division of simple quantities, the course to be pursued is,*

To divide the coefficient of the dividend by that of the divisor;

To suppress in the dividend the letters which are common to it and the divisor, when they have the same exponent; and when the exponent is not the same, to subtract the exponent of the divisor from that of the dividend, the remainder being the exponent to be affixed to the letter in the quotient;

To write in the quotient the letters of the dividend which are not in the divisor.

37. If we apply the rule now given for obtaining the exponent of the letters of the quotient, to a letter which has the same

exponent in the dividend and divisor, we shall find zero to be the exponent which it ought to have in the quotient; a^3 divided by a^3, for example, gives a^0. To understand what is the import of such an expression, it is necessary to go back to its origin and to consider, that if we represent the quotient arising from the division of a quantity by itself, it ought to answer to unity, which expresses how many times any quantity is contained in itself. It follows from this, that *the expression* a^0 *is a symbol equivalent to unity, and may consequently be represented by* 1. We may then omit writing the letters which have zero for their exponent, since each of them signifies nothing but unity. Thus $a^3 b c^2$ divided by $a^2 b c^2$, gives $a^1 b^0 c^0$, which becomes a, as is very evident by suppressing the common factors of the dividend and divisor.

We see by this, that the proposition, *every quantity which has zero for its exponent, is equal to* 1, is nothing, properly speaking, but the explanation of a conclusion to which we are brought by the common manner of writing the powers of quantities by exponents.

In order that the division may be performed, it is necessary, 1. that the divisor should have no letter which is not found in the dividend; 2. that the exponent of any letter in the divisor should not exceed that of the same letter in the dividend; 3. that the coefficient of the divisor should exactly divide that of the dividend.

38. When these conditions do not exist, the division can only be indicated in the manner pointed out in the 2d article. Still we should endeavour to simplify the fraction by suppressing such factors, as are common to the dividend and divisor, if there are any such; for (*Arith.* 57) it is manifest, that the theory of arithmetical fractions rests upon principles which are independent of every particular value of their terms, and which would apply to fractions represented by letters, as well as to those which are represented by numbers.

According to these principles, *we in the first place suppress the numerical factors common to the dividend and divisor, and then the letters which are common to the dividend and divisor, and which have the same exponent in each. When the exponent is not the same in each, we subtract the less from the greater, and affix the remainder, as the exponent to the letter, which is written only in that term of the fraction which has the highest exponent.*

The following example will illustrate this rule.

Let $48\,a^3\,b^5\,c^2\,d$ be divided by $64\,a^3\,b^3\,c^4\,e$; the quotient can only be indicated in the form of a fraction

$$\frac{48\,a^3\,b^5\,c^2\,d}{64\,a^3\,b^3\,c^4\,e}.$$

But the coefficients 48 and 64 being divisible by 16, by suppressing this common factor, the coefficient of the numerator becomes 3, and that of the denominator 4. The letter a having the same exponent 3 in the two terms of the fraction, it follows that a^3 is a factor common to the dividend and divisor, and may consequently be suppressed.

To find the number of factors b common to the two terms of the fraction, we must divide the higher b^5 by the lower b^3, according to the rule above given, and the quotient b^2 shows, that $b^5 = b^3 \times b^2$. Suppressing then the common factor b^3, there will remain in the numerator the factor b^2.

With respect to the letter c, the higher factor being c^4 of the denominator, if we divide it by c^2 we shall decompose it into $c^2 \times c^2$; and by suppressing the factor c^2 common to the two terms, this letter disappears from the numerator, but will remain in the denominator with the exponent 2.

Finally, the letters d and e, will remain in their respective places, since in the state in which they are, they indicate no factor common to both.

By these several operations the proposed fraction is reduced to

$$\frac{3\,b^2\,d}{4\,c^2\,e};$$

and it is the most simple expression of the quotient, except we give numerical values to the letters, in which case it might be further reduced by cancelling the common factors as before.

39. It ought to be remarked, that if all the factors of the dividend enter into the divisor, which besides contains others peculiar to it, it is necessary after suppressing the former to put unity in the place of the dividend as the numerator of the fraction. In this case indeed we may suppress all the terms of the numerator, or in other words, divide the two terms of the fraction by the numerator; but this being divided by itself must give unity for the quotient, which becomes the new numerator.

Suppose for example the fraction

$$\frac{4\,a^2\,b\,c}{12\,a^2\,b^3\,c\,d},$$

the factors 12, a^2, b^3 and c may be divided respectively by the factors 4, a^2, b and c, or we may divide the two terms of the fraction by the numerator $4\,a^2\,b\,c$. Now the quantity $4\,a^2\,b\,c$ divided by itself gives 1 for the quotient, and the quantity $12\,a^2\,b^3\,cd$ divided by the first, gives by the above rules $3\,b^2\,d$; the new fraction then is

$$\frac{1}{3\,b^2\,d},$$

40. It follows from the rules of multiplication, that when a compound quantity is multiplied by a simple quantity, this last becomes a factor common to all the terms of the former. We may make use of this observation to simplify fractions of which the numerator and denominator are polynomials having factors that are common to all their terms.

Let their be the expression

$$\frac{6\,a^4 - 3\,a^2\,bc + 12\,a^2\,c^2}{9\,a^2\,b - 15\,a^2\,c + 24\,a^3};$$

by examining the quantity $6\,a^4 - 3\,a^2\,b\,c + 12\,a^2\,c^2$, we see that the factor a^2 is common to all the terms, since $a^4 = a^2 \times a^2$, and that, besides, 6, 3 and 12 are all divisible by 3; so that,

$6\,a^4 - 3\,a^2\,bc + 12\,a^2\,c^2 = 2\,a^2 \times 3\,a^2 - b\,c \times 3\,a^2 + 4\,c^2 \times 3\,a^2$.

Also the denominator has for a common factor $3\,a^2$; for the factors a^2 and 3 enter into all the terms, and we have

$9\,a^2\,b - 15\,a^2\,c + 24\,a^3 = 3\,b \times 3\,a^2 - 5\,c \times 3\,a^2 + 8\,a \times 3\,a^2$.

Suppressing therefore the $3\,a^2$ as often in the numerator as in the denominator, the proposed fraction will become

$$\frac{2\,a^2 - b\,c + 4\,c^2}{3\,b - 5\,c + 8\,a}.$$

41. I pass now to the case where the numerator and denominator are both compound, and in which one cannot perceive at first whether the divisor is or is not a factor of the dividend.

As the divisor multiplied by the quotient must produce the dividend, it is necessary that this last should contain all the several products of each term of the divisor by each term of the quotient: and if we could find the products arising from each particular term of the divisor; by dividing them by this term, which is known, we should obtain those of the quotient, after the same manner as in arithmetic we discover all the figures of the quotient by dividing successively by the divisor the numbers, which we regard as the several products of this divisor by the different fig-

ures of the quotient. But in numbers the several products present themselves in order, beginning with the units at the last place on the left, on account of the subordination established between the units of each figure of the dividend according to the rank which they hold. But as this is not the case in algebra, we supply the want of such an arrangement by disposing all the terms of the dividend and divisor in the order of the exponents of the power of the same letter, beginning with the highest and proceeding from left to right, as may be seen with reference to the letter a in the quantities

$$5a^7 - 22a^6b + 12a^5b^2 - 6a^4b^3 - 4a^3b^4 + 8a^2b^5,$$

$$5a^4 - 2a^3b + 4a^2b^2,$$

of which one is the product and the other the multiplicand in the example of art. 32. This is called *arranging* the proposed quantities.

When they are thus disposed, it is evident, that whatever be the factor by which it is necessary to multiply the second to obtain the first, the term $5a^7$, with which this begins, results from the multiplication of $5a^4$, with which the other begins, by the term in the factor sought, in which a has the highest exponent, and which takes the first place in this factor when the terms of it are arranged with reference to the letter a. By dividing then the simple quantity $5a^7$ by the simple quantity $5a^4$, the quotient a^3 will be the first term of the factor sought. Now as the entire product ought by the rules of multiplication to contain the several particular products arising from the multiplication of the whole multiplicand by each term of the multiplier, it follows that the quantity here taken for the dividend, ought to contain the products of all the terms of the divisor, $5a^4 - 2a^3b + 4a^2b^2$, by the first term of the quotient a^3; and consequently, if we subtract from the dividend these products, which are $5a^7 - 2a^6b + 4a^5b^2$, the remainder $-20a^6b + 8a^5b^2 - 6a^4b^3 - 4a^3b^4 + 8a^2b^5$ will contain only those, which result from the multiplication of the divisor by the second, third, &c. terms of the quotient.

The remainder then may be considered as a part of the dividend, and its first term, in which a has the highest exponent, cannot be obtained, otherwise than by the multiplication of the first term of the divisor by the second term of the quotient. But the first term of this part of the dividend having the sign $-$, it is necessary to assign that which is to be prefixed to the corresponding *term of the quotient*. This is easily done by the first

rule art. 31, for the quantity $-20\,a^6\,b$, being regarded as a part of the product, having a sign contrary to that of the multiplicand $5\,a^4$, it follows that the multiplier must have the sign $-$. Division then being performed upon the simple quantities, $-20\,a^6\,b$ and $5\,a^4$, gives $-4\,a^2\,b$ for the second term of the quotient. If now we multiply this by all the terms in the divisor, and subtract the product from the partial dividend, the remainder $+10\,a^4\,b^3 - 4\,a^3\,b^4 + 8\,a^2\,b^5$ will contain only the products of the third &c. terms of the quotient.

Regarding this remainder as a new dividend, its first term $10\,a^4\,b^3$ must be the product of the first term of the divisor by the third of the quotient, and consequently this last is obtained by dividing the simple quantities, $10\,a^4\,b^3$ and $5\,a^4$ the one by the other. The quotient $2\,b^3$ being multiplied by the whole of the divisor furnishes products, the subtraction of which exhausting the remaining dividend, proves that the quotient has only three terms.

If the question had been such as to require a greater number of terms, they might evidently have been found like the preceding, and if, as we have supposed, the dividend has the divisor for a factor, the subtraction of the product of this divisor by the last term of the quotient ought always to exhaust the corresponding dividend.

42. To facilitate the practice of the above rules;

1. *We dispose the dividend and divisor, as for the division of numbers, by arranging them with reference to some letter, that is, by writing the terms in the order of the exponents of this letter, beginning with the highest;*

2. *We divide the first term of the dividend by the first term of the divisor, and write the result in the place of the quotient;*

3. *We multiply the whole divisor by the term of the quotient just found, subtract it from the dividend, and reduce similar terms.*

4. *We regard this remainder as a new dividend, the first term of which we divide by the first term of the divisor, and write the result as the second term of the quotient, and continue the operation till all the terms of the dividend are exhausted.*

Recollecting that a product has the same sign as the multiplicand when the multiplier has the sign $+$, and that it has in the contrary case the sign $-$ (31), we infer that, *when the term of the dividend and the first term of the divisor have the same sign, the*

quotient ought to have the sign +, and if they have contrary signs, the quotient ought to have the sign —; this is the rule for the signs.

The individual parts of the operation are performed by the rule for the division of simple quantities.

We divide the coefficient of the dividend by that of the divisor; this is the rule for the coefficients.

We write in the quotient the letters common to the dividend and divisor with an exponent equal to the difference of the exponents of these letters in the two terms, and the letters which belong only to the dividend; these are the rules for the letters and exponents.

43. To apply these rules to the quantities,

$$5\,a^7 - 22\,a^6\,b + 12\,a^5\,b^2 - 6\,a^4\,b^3 - 4\,a^3\,b^4 + 8\,a^2\,b^5,$$
$$5\,a^4 - 2\,a^3\,b + 4\,a^2\,b^2,$$

which have been employed as an example above, we place them as we place the dividend and divisor in arithmetic.

$$\begin{array}{l|l}
\textit{Dividend.} & \textit{Divisor.} \\
5a^7 - 22a^6b + 12a^5b^2 - 6a^4b^3 - 4a^3b^4 + 8a^2b^5 & 5a^4 - 2a^3b + 4a^2b^2 \\
\cline{2-2}
-5a^7 + 2a^6b - 4a^5b^2 & \textit{Quotient.} \\
 & a^3 - 4a^2\,b + 2b^2 \\
\cline{1-1}
\text{Rem.} -20a^6b + 8a^5b^2 - 6a^4b^3 - 4a^3b^4 + 8a^2b^5 & \\
\quad + 20a^6b - 8a^5b^2 + 16a^4b^3 & \\
\cline{1-1}
\quad \text{rem.} \qquad + 10a^4b^3 - 4a^3b^4 + 8a^2b^5 & \\
\qquad\qquad\quad - 10a^4b^3 + 4a^3b^4 - 8a^2b^5 & \\
\cline{1-1}
\qquad\qquad\qquad\qquad 0. &
\end{array}$$

The sign of the first term $5\,a^7$ of the dividend being the same as that of $5\,a^4$, the first term of the divisor, the sign of the quotient must be +, but as it is the first term, the sign is omitted.

By dividing $5\,a^7$ by $5\,a^4$, we have for the quotient a^3, which we write under the divisor.

Multiplying successively the three terms of the divisor by the first term a^3 of the quotient, and writing the products under the corresponding terms of the dividend, the signs being changed to denote their subtraction (20), we have the quantity

$$-5\,a^7 + 2a^6\,b - 4\,a^5\,b^2,$$

which with the dividend being reduced, we obtain for a remainder

$$-20\,a^6\,b + 8\,a^5\,b^2 - 6\,a^4\,b^3 - 4\,a^3\,b^4 + 8\,a^2\,b^5.$$

By continuing the division with this remainder, the first term $-20\,a^6\,b$, divided by $5\,a^4$, will give for a quotient $4\,a^2\,b$, this quotient having the sign —, as the dividend and divisor have differ-

ent signs. Multiplying it by all the terms of the divisor and changing the signs, we obtain the quantity

$$20\,a^6\,b - 8\,a^5\,b^2 + 16\,a^4\,b^3,$$

which taken with the dividend and reduced, gives for a remainder

$$+\,10\,a^4\,b^3 - 4\,a^3\,b^4 + 8\,a^2\,b^5.$$

Dividing the first term of this new dividend, $10\,a^4\,b^3$, by the first term, $5\,a^4$, of the divisor, and multiplying the whole divisor by the result $+\,2\,b^3$, writing the products under the dividend, the signs being changed and making the reduction, we find that nothing remains, which shows that $+\,2\,b^3$ is the last term of the quotient sought. The quotient therefore has for its expression $a^3 - 4\,a^2\,b + 2\,b^3$.

44. It is proper to remark here, that in division, the multiplication of the different terms of the quotient by the divisor often produces terms that are not to be found in the dividend, and which it is necessary to divide by the first term of the divisor. These terms are such as destroy themselves, since the dividend has been formed by the multiplication of the two factors the quotient and the divisor. See a remarkable example of these reductions;

Let $a^3 - b^3$ be divided by $a - b$.

Division.

$$\begin{array}{rrr|l}
a^3 & & -\,b^3 & a - b \\
\cline{4-4}
-\,a^3 + a^2\,b & & & a^2 + a\,b + b^2 \\
\cline{1-3}
+\,a^2\,b & & -\,b^3 & \\
-\,a^2\,b & +\,a\,b^2 & & \\
\cline{1-3}
& +\,a\,b^2 & -\,b^3 & \\
& -\,a\,b^2 & +\,b^3 & \\
\cline{2-3}
& 0 & 0 &
\end{array}$$

Multiplication.

$$\begin{array}{rrrr}
& & a - b & \\
& & a^2 + a\,b + b^2 & \\
\hline
& a^3 & -\,a^2\,b & \\
& & +\,a^2\,b & -\,a\,b^2 \\
& & & +\,a\,b^2 - b^3 \\
\hline
\text{Result} & a^3 & & -\,b^3.
\end{array}$$

The first term a^3 of the dividend, divided by the first term a of the divisor, gives for the quotient a^2; multiplying this quotient by the divisor, and changing the signs of the products, we have $-\,a^3 + a^2\,b$; the first term $-\,a^3$ destroys the first term of the dividend, but there remains the term $a^2\,b$, which is not found at first in the dividend. As it contains the letter a, we can divide it by the first term of the divisor, and obtain $+\,a\,b$. Multiplying this quotient by the divisor, and changing the signs of the products we have $-\,a^2\,b + a\,b^2$; the term $-\,a^2\,b$ cancels the one above it, but there remains the term $+\,ab^2$, which is not in the dividend. This being divided by a gives for the quotient $+\,b^2$; *multiplying* this quotient by the divisor and changing the signs,

we have $-ab^2+b^3$; the first term $-ab^2$ destroys the first term of the dividend, and the second $+b^3$ destroys the other $-b^3$.

The mechanical part of the operation will be better understood, if we look for a moment at the multiplication of the quotient a^2+ab+b^2 by the divisor $a-b$. We see that all the terms reproduced in the process of dividing are those which destroy each other in the result of the multiplication.

45. It sometimes happens that the quantity with reference to which the arrangement is made, has the same power in several terms both of the dividend and divisor. In this case, the terms should be written in the same column, one under the other, the remaining ones being disposed with reference to another letter.

Let there be

$$-a^4b^2+b^2c^4-a^2c^4-a^6+2a^4c^2+b^6+2b^4c^2+a^2b^4,$$

to be divided by $a^2-b^2-c^2$.

Arranging the first of these quantities with reference to the letter a, we place in the same column the terms $-a^4b^2$ and $+2a^4c^2$, in another, the terms $+a^2b^4$ and $-a^2c^4$; and in the last column the three terms $+b^6$, $+2b^4c^2$, $+b^2c^4$, disposing them with reference to the letter b, as may be seen in the next page.

The first term a^6 of the dividend being divided by the first term a^2 of the divisor, gives for the first term of the quotient $-a^4$; forming the products of this quotient by all the terms of the divisor, changing the signs of the products in order to subtract them from the dividend, and placing in the same column the terms containing the same power of a, we have, after the reduction of similar terms, the first remainder, which we take for the second dividend.

The first term $-2a^4b^2$ of this new dividend, being divided by a^2, gives for the second term of the quotient $-2a^2b^2$; forming the products of this quotient by all the terms of the divisor, changing the signs of the products to indicate their subtraction from the dividend, and placing in the same column the terms containing the same power of a, we have after the reduction of similar terms, the second remainder, which we take for the third dividend.

The operation being continued in the same manner with the second remainder and the following ones, we shall have three

terms in the quotient. The last being multiplied by all the terms of the divisor, furnishes products which being subtracted from the fourth remainder exhaust it entirely. As the division admits of being exactly performed, it follows, that the divisor is a factor of the dividend

$$\begin{array}{l|l}
\begin{array}{l}
-a^6 - a^4b^2 + a^2b^4 + b^6 \\
\quad + 2a^4c^2 - a^2c^4 + 2b^4c^2 \\
\qquad\qquad\qquad + b^2c^4 \\
+a^6 - a^4b^2 \\
\quad - a^4c^2
\end{array} &
\begin{array}{l}
a^2 - b^2 - c^2 \\ \hline
-a^4 - 2a^2b^2 - b^4 \\
\quad + a^2c^2 - b^2c^2
\end{array}
\end{array}$$

$$\begin{array}{ll}
\text{1st. rem.} & -2a^4b^2 + a^2b^4 + b^6 \\
& + a^4c^2 - a^2c^4 + 2b^4c^2 \\
& \qquad\qquad + b^2c^4 \\
& +2a^4b^2 - 2a^2b^4 \\
& \qquad - 2a^2b^2c^2 \\ \hline
\text{2d. rem.} & + a^4c^2 - a^2b^4 + b^6 \\
& \qquad - 2a^2b^2c^2 + 2b^4c^2 \\
& \qquad - a^2c^4 + b^2c^4 \\
& -a^4c^2 + a^2b^2c^2 \\
& \qquad + a^2c^4 \\ \hline
\text{3d. rem.} & -a^2b^4 + b^6 \\
& -a^2b^2c^2 + 2b^4c^2 \\
& \qquad + b^2c^4 \\
& +a^2b^4 - b^6 \\
& \qquad - b^4c^2 \\ \hline
\text{4th. rem.} & -a^2b^2c^2 + b^4c^2 \\
& \qquad + b^2c^4 \\
& +a^2b^2c^2 - b^4c^2 \\
& \qquad - b^2c^4 \\ \hline
& \quad 0 \qquad 0
\end{array}$$

46. The form under which a quantity appears, will sometimes immediately suggest the factors into which it may be decomposed. If we have, for example,

$$8a^6 - 4a^3b^2 + 4a^3 + 2a^3 - b^2 + 1,$$

to be divided by $2a^3 - b^2 + 1$; as the divisor forms the three last terms of the dividend, it is only necessary to see if it is a factor of the three first; but these have obviouslyfor a common factor $4a^3$, for $8a^6 - 4a^3b^2 + 4a^3 = 4a^3\ (2a^3 - b^2 + 1)$.
The dividend then may be represented by

$$4a^3(2a^3-b^2+1)+2a^3-b^2+1,$$

or
$$(2a^3-b^2+1)(4a^3+1).$$

The division is performed at once by suppressing the factor $2a^3-b^2+1$, equal to the divisor, and the quotient will be $4a^3+1$.

After a little practice, methods of this kind will readily occur, by which algebraic operations are abridged.

By frequent exercise in examples of this kind, the resolution of a quantity into its factors is at length easily performed; and it is often rendered very conspicuous, when, instead of performing the operations represented, they are only indicated.

Of algebraic fractions.

47. When we apply the rules of algebraic division to quantities, of which the one is not a factor of the other, we perceive the impossibility of performing it, since in the course of the operation we arrive at a remainder, the first term of which is not divisible by that of the divisor. See an example;

$$\begin{array}{rl|l}
 & a^3+a^2b+2b^3 & a^2+b^2 \\ \cline{3-3}
 & -a^3-ab^2 & a+b \\ \cline{2-2}
\text{1st. rem.} & a^2b-ab^2+2b^3 & \\
 & -a^2b-b^3 & \\ \cline{2-2}
\text{2d. rem.} & -ab^2+b^3 &
\end{array}$$

The first term $-ab^2$ of the second remainder cannot be divided by a^2, the first term of the divisor, so that the process is arrested at this point. We can however, as in arithmetic, annex to the quotient $a+b$ the fraction $\frac{-ab^2+b^3}{a^2+b^2}$, having the remainder for the numerator, and the divisor for the denominator; and the quotient will be

$$a+b+\frac{b^3-ab^2}{a^2+b^2}.$$

It is evident, that *the division must cease, when we come to a remainder, the first term of which does not contain the letter with reference to which the terms are arranged, or to a power inferior to that of the same letter in the first term of the divisor.*

48. When the algebraic division of the two quantities cannot be performed, the expression of the quotient remains indicated under the form of a fraction, having the dividend for the numerator, and the divisor for the denominator; and to abridge it as

much as possible, we should see if the dividend and divisor have not common factors, which may be cancelled (38). But when the terms of the fraction are polynomials, the common factors are not so easily found as when they are simple quantities. They are in general to be sought by a method analogous to that, which is given in arithmetic for finding the *greatest common divisor* of two numbers.

We cannot assign the relative magnitudes of algebraic expressions, as we do not give values to the letters which they contain; the denomination of *greatest common divisor* therefore, applied to these expressions, ought not to be taken altogether in the same sense as in arithmetic.

In algebra we are to understand by the *greatest common divisor* of two expressions, that which contains the most factors in all its terms, or which is of the highest degree (27). Its determination rests, as in arithmetic, upon this principle; *Every common divisor to two quantities must divide the remainder after their division.*

The demonstration given in arithmetic (art. 61) is rendered clearer by employing algebraic symbols. If we represent the common divisor by D, the two quantities proposed might be expressed by the products $A\ D$ and $B\ D$, formed from the common divisor and the factor by which it is multiplied in each of the quantities. This being supposed, if Q stands for the entire quotient, and R for the remainder resulting from the division of $A\ D$ by $B\ D$, we have $A\ D = B\ D \times Q + R$ (*Arith.* 61); dividing now the two members of the equation by D, we obtain

$$A = B\ Q + \frac{R}{D};$$

and since the first member, which in this case must be composed of the same terms, as the second, is entire, it must follow, that $\frac{R}{D}$ is reduced to an expression without a divisor, that is to say, that R is divisible by D.

According to this principle, *we begin, as in arithmetic, by inquiring whether one of the quantities is not itself the divisor of the other; if the division cannot be exactly performed, we divide the first divisor by the remainder, and so on; and that remainder, which will exactly divide the preceding, will be the greatest common divisor of the two quantites proposed.* But it will be necessary in

the divisions indicated, to have regard to what belongs to the nature of algebraic quantities.

We are not, in the first place, to seek a common divisor of two algebraic quantities, except when they have common letters; and we must select from them a letter, with reference to which the proposed expressions are to be arranged, and that is to be taken for the dividend in which this letter has the highest exponent, the other being the divisor.

Let there be the two quantities

$$3a^3 - 3a^2 b + a b^2 - b^3,$$
$$4a^2 b - 5 a b^2 + b^3,$$

which are already arranged with reference to the letter a; we take the first for the dividend and the second for the divisor. A difficulty immediately presents itself, which we do not meet with in numbers, and this is, that the first term of the divisor will not exactly divide the first term of the dividend, on account of the factors 4 and b in the one, which are not in the other. But the letter b being common to all the terms of the divisor and not to those of the dividend, it follows (40) that b is a factor of the divisor, and that it is not of the dividend. Now every divisor common to two quantities can consist only of factors which are common to the one and to the other; if then there be such a divisor with respect to the two quantities proposed, it is to be looked for among the factors of the quantity $4a^2 - 5ab + b^2$, which remains of the quantity $4a^2 b - 5ab^2 + b^3$, after suppressing b; so that the question reduces itself to finding the greatest common divisor of the two quantities

$$3a^3 - 3a^2 b + a b^2 - b^3,$$
$$4a^2 - 5ab + b^2.$$

For the same reason that we may cancel in one of the proposed quantities the factor b which is not in the other, we may likewise introduce into this a new factor, provided it is not a factor of the first. By this step, the greatest common divisor, which can consist only of terms common to both, will not be affected. Availing myself of this principle, I multiply the quantity $3a^3 - 3a^2 b + a b^2 - b^3$ by 4, which is not a factor of the quantity $4a^2 - 5ab + b^2$, in order to render the first term of the one divisible by the first term of the other.

I shall thus have for the dividend the quantity

$$12a^3 - 12a^2 b + 4ab^2 - 4b^3,$$

for *the divisor the quantity*

$$4a^2 - 5ab + b^2,$$

and the quotient will be $3a$.

Multiplying the divisor by this quotient, and subtracting the product from the dividend, I have for a remainder

$$3a^2 b + ab^2 - 4b^3,$$

a quantity which, according to the principle stated at the commencement of this article, must have with $4a^2 - 5ab + b^2$, the same greatest common divisor as the first.

Profiting by the remarks made above, I suppress the factor b, common to all the terms of this remainder, and multiply it by 4 in order to render the first term divisible by that of the divisor; I have then for a dividend the quantity

$$12a^2 + 4ab - 16b^2,$$

and for a divisor the quantity

$$4a^2 - 5ab + b^2;$$

and the quotient thence arising is 3.

Multiplying the divisor by the quotient, and subtracting the product from the dividend, we obtain the remainder

$$19ab - 19b^2,$$

and the question is reduced to finding the greatest common divisor to this quantity, and

$$4a^2 - 5ab + b^2.$$

But the letter a, with reference to which the division is made, not being in the remainder, except of the first degree, while it is of the second degree in the divisor, it is this which must be taken for the dividend, and the remainder must be made the divisor.

Before beginning this new division I expunge from the divisor $19ab - 19b^2$, the factor $19b$ common to both the terms, and which is not a factor of the dividend; I have then for a dividend the quantity

$$4a^2 - 5ab + b^2,$$

and for a divisor

$$a - b.$$

The division leaves no remainder; so that $a - b$ is the greatest common divisor required.

By retracing these steps, we may prove *à posteriori*, that the quantity $a - b$ must exactly divide the two quantities proposed, and that it is the most compounded of those which will do it. In dividing by $a - b$ the two quantities proposed

$$3a^3 - 3a^2 b + ab^2 - b^3,\quad 4a^2 b - 5ab^2 + b^3,$$

we resolve them as follows;

$$(3a^2+b^2)(a-b), \quad (4ab-b^2)(a-b).$$

49. When the quantity, which we take for a divisor, contains several terms having the letter, with reference to which the arrangement is made, of the same degree, there are precautions to be used, without which the operation would not terminate. See an example of this.

Let there be the quantities

$$a^2b+ac^2-d^3, \quad ab-ac+d^2;$$

if we make the preparation as for common division

$$\begin{array}{r|l} a^2b+ac^2-d^3 & ab-ac+d^2 \\ \cline{2-2} -a^2b+a^2c-ad^2 & a \\ \hline \text{Rem. } a^2c+ac^2-ad^2-d^3, & \end{array}$$

by dividing first a^2b by ab, we have for the quotient a; multiplying the divisor by this quotient, and subtracting the products from the dividend, the remainder will contain a new term, in which a will be of the second degree, namely, a^2c, arising from the product of $-ac$ by a. Thus no progress has been made; for by taking the remainder

$$a^2c+ac^2-ad^2-d^3$$

for a dividend, and multipling by b to render the division possible by ab, we have

$$\begin{array}{l|l} a^2bc+abc^2-abd^2-bd^3 & ab-ac+d^2 \\ \cline{2-2} -a^2bc+a^2c^2-acd^2 & ac \\ \hline \text{rem. } a^2c^2+abc^2-acd^2-abd^2-bd^3, & \end{array}$$

and the term $-ac$ produces still a term a^2c^2, in which a is of the second degree.

To avoid this inconvenience, it must be observed, that the divisor $ab-ac+d^2=a(b-c)+d^2$, by uniting the terms $ab-ac$ in one; and for the sake of shortening the operation, making $b-c=m$, we have for a divisor $am+d^2$; but then the whole dividend must be multiplied by the factor m, to make a new dividend, the first term of which may be divided by am, the first term of the divisor; the operation then becomes

$$\begin{array}{rl|l} & a^2bm+ac^2m-d^3m & am+d^2 \\ \cline{3-3} & a^2bm-abd^2 & ab+c^2 \\ \cline{2-2} \text{1st. rem.} & -abd^2+ac^2m-d^3m & \\ & -ac^2m-c^2d^2 & \\ \cline{2-2} \text{2d. rem.} & -abd^2-c^2d^2-d^3m & \end{array}$$

The terms involving a^2 now disappear from the dividend, and there remain only the terms which have the first power of a. To make these disappear we first divide the term $a c^2 m$ by $a m$, and it gives for a quotient c^2; multiplying the divisor by this quotient, and subtracting the products from the dividend, we obtain the second remainder. Taking this second remainder for a new dividend, and suppressing the factor d^2, which is not a factor of the divisor, we have

$$-ab-c^2-dm,$$

which being multiplied anew by m, becomes

$$\begin{array}{r|l} -abm-c^2m-dm^2 & am+d^2 \\ \cline{2-2} +abm+bd^2 & -b \\ \cline{1-1} \text{Rem.}\quad +bd^2-c^2m-dm^2 & \end{array}$$

The remainder $b\,d^2-c^2m-d\,m^2$ of this last division, not involving a, it follows that if the proposed quantities have a common divisor, it is independent of the letter a.

Having arrived at this point, we can continue the division no longer with reference to the letter a; but it will be observed, that if there be a common divisor, independent of a, to the quantities $b\,d^2-c^2\,m-d\,m^2$ and $a\,m+d^2$, it must divide separately the two parts $a\,m$ and d^2 of the divisor; for if a quantity is arranged with reference to the powers of the letter a, every divisor of this quantity, independent of a, must divide separately the quantities multiplied by the different powers of this letter.

To be convinced of this we need only observe, that in this case, each of the quantities proposed must be the product of a quantity depending on a, and of the common divisor which does not depend upon it. Now if we have, for example, the expression

$$A\,a^4+B\,a^3+C\,a^2+D\,a+E,$$

in which the letters A, B, C, D, E, designate any quantities whatever, independent of a, and it be multiplied by a quantity also independent of a, the product

$$M\,A\,a^4+M\,B\,a^3+M\,C\,a^2+M\,D\,a+M\,E,$$

arranged with reference to a, will contain still the same powers of a as before; but the coefficient of each of these powers will be a multiplier of M.

This being supposed, if we restore the quantity $(b-c)$ in the place of m, we have the quantities

$$b\,d^2 - c^2\,(b-c) - d\,(b-c)^2,$$
$$a\,(b-c) + d^2\,;$$
and it is evident, that $b-c$ and d^2 have no common factor; the two quantities then under consideration have not a common divisor.

If it were not evident by mere inspection, that there is no common divisor between $b-c$ and d^2, it would be necessary to seek their greatest common divisor by arranging them with reference to the same letter, and then to see if it would not also divide the quantity
$$b\,d^2 - c^2\,(b-c) - d\,(b-c)^2.$$

50. Instead of putting off to the end of the operation, the investigation of the greatest common divisor independent of the letter with reference to which the quantities are arranged, it is less trouble to seek it at first, because, for the most part, the operation becomes more complicated at each step as we advance, and the process is rendered more difficult.

Let there be, for example, the quantities
$$a^4\,b^2 + a^3\,b^3 + b^4\,c^2 - a^4\,c^2 - a^3\,b\,c^2 - b^2\,c^4,$$
$$a^2\,b + a\,b^2 + b^3 - a^2\,c - a\,b\,c - b^2\,c\,;$$
having arranged them with reference to the letter a we have
$$(b^2 - c^2)\,a^4 + (b^3 - b\,c^2)\,a^3 + b^4\,c^2 - b^2\,c^4,$$
$$(b-c)\,a^2 + (b^2 - b\,c)\,a + b^3 - b^2 c,$$
I observe in the first place, that if they have a common divisor which is independent of a, it must divide each of the quantities multiplied by the different powers of a (49), as well as the quantities $b^4\,c^2 - b^2\,c^4$ and $b^3 - b^2\,c$, which do not contain this letter.

The question is reduced then to finding the common divisors of the two quantities $b^2 - c^2$ and $b - c$, and determining whether among these divisors there is to be found one which will divide at the same time
$$b^3 - b\,c^2 \text{ and } b^2 - b\,c, \quad b^4\,c^2 - b^2\,c^4 \text{ and } b^3 - b^2\,c.$$
Dividing $b^2 - c^2$ by $b - c$, we find an exact quotient $b + c$; $b - c$ then is a common divisor of the quantities $b^2 - c^2$ and $b - c$, which evidently admit of no other, since the quantity $b - c$ is divisible only by itself and by unity. We must now see whether $b - c$ will divide the other quantities referred to above, or whether it will divide the two quantities proposed; it is found that it will, and it gives
$$(b+c)\,a^4 + (b^2 + b\,c)\,a^3 + b^3\,c^2 + b^2\,c^3,$$
$$a^2 + b\,a + b^2.$$

To bring these last expressions to the greatest degree of simplicity, we should see if the first is not divisible by $b+c$; it appears upon trial that it is, and we have only to find a common divisor to the quantities

$$a^4+ba^3+b^2c^2,$$
$$a^2+ba\ +b^2.$$

By proceeding with these as the rule prescribes, we come after the second division to a remainder containing the letter a of the first power only; and as this remainder is not the common divisor, we conclude that the letter a does not make a part of the common divisor sought, which therefore can consist only of the factor $b-c$.

If beside this common divisor, another had been found involving the quantity a, it would have been necessary to multiply these two divisors together to obtain the common divisor sought.

These remarks will enable the learner, after a little practice in analysis, to find in every case the greatest common divisor. He will determine without difficulty that the quantities

$$6a^5+15a^4b-4a^3c^2-10a^2bc^2,$$
$$9a^3b-27a^2bc-6abc^2+18bc^3,$$

have for their greatest common divisor the quantity $3a^2-2c^2$.

51. The four *fundamental operations*, addition, subtraction, multiplication and division, we perform in algebra as in arithmetic, observing merely to proceed, in the operations prescribed by the rules of arithmetic, according to the methods given for algebraic quantities. I shall, therefore, merely suggest these methods, giving an example of the application of each. I shall begin as I did in arithmetic, with the multiplication and division of fractions, as they require no preparatory transformations.

1. For multiplication, we have

$$\frac{a}{b}\times c=\frac{ac}{b}\ (\textit{Arith.}\ 53),$$
$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}\ (\textit{Arith.}\ 70).$$

2. For division,

$$\frac{a}{b}\text{ divided by }c\text{, gives }\frac{a}{bc}\text{ or }\frac{a}{b}\times\frac{1}{c}\ (\textit{Arith.}\ 54,\ 70)$$
$$\frac{a}{b}\text{ divided by }\frac{c}{d}\text{, gives }\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}\ (\textit{Arith.}\ 73).$$

3. The fractions $\frac{a}{b}$, $\frac{c}{d}$, being reduced to the same denominator, become respectively

$$\frac{ad}{bd},\ \frac{bc}{bd}\ (\textit{Arith.}\ 79).$$

The fractions,

$$\frac{a}{b},\ \frac{c}{d},\ \frac{e}{f},\ \frac{g}{h},$$

by the same reduction become respectively

$$\frac{adfh}{bdfh},\ \frac{cbfh}{bdfh},\ \frac{ebdh}{bdfh},\ \frac{gbdf}{bdfh}.$$

52. I have given in art. 79 of arithmetic, a process for obtaining in certain cases a denominator more simple, than that which results from the general rule; it may be much simplified by means of algebraic symbols, as we shall see.

If, for example, we have the two fractions $\frac{a}{bc}$, $\frac{d}{bf}$, it is easy to see that the two denominators would be the same, if f were a factor of the first, and c a factor of the second; we multiply then the two terms of the first fraction by f, and the two terms of the second by c, which gives $\frac{af}{bcf}$ and $\frac{cd}{bcf}$, more simple than $\frac{abf}{bbcf}$ and $\frac{bcd}{bbcf}$, obtained by multiplying by the original denominators.

In general, *to form the common denominator, we collect into one product all the different factors raised to the highest power found in the denominators of the proposed fractions, and it remains only to multiply the numerator of each fraction by the factors of this product, which are wanting in the denominator of the fraction.*

Having, for example, the fractions $\frac{a}{b^2c}$, $\frac{d}{bf}$ and $\frac{e}{cg}$, I form the product b^2cfg; I multiply the numerator of the first fraction by fg, that of the second bcg, that of the third by b^2f, and I obtain

$$\frac{afg}{b^2cfg},\qquad \frac{bcdg}{b^2cfg},\qquad \frac{b^2ef}{b^2cfg}.$$

53. The sum of the fractions

$$\frac{a}{d},\qquad \frac{b}{d},\qquad \frac{c}{d},$$

which have the same denominator, or

$$\frac{a}{d}+\frac{b}{d}+\frac{c}{d}=\frac{a+b+c}{d}\ (\textit{Arith.}\ 80).$$

8

The difference of the fractions

$$\frac{a}{d} \text{ and } \frac{b}{d},$$

which have the same denominator, or

$$\frac{a}{d}-\frac{b}{d}=\frac{a-b}{d}.$$

The whole of a added to the fraction $\frac{b}{c}$, or the expression

$$a+\frac{b}{c}=\frac{ac}{c}+\frac{b}{c}=\frac{ac+b}{c} \text{ (Arith. 81).}$$

Also the expression

$$a-\frac{b}{c}=\frac{ac}{c}-\frac{b}{c}=\frac{ac-b}{c}.$$

Reciprocally

the expression $\frac{ac+b}{c}=\frac{ac}{c}+\frac{b}{c}=a+\frac{b}{c}$,

the expression $\frac{ac-b}{c}=\frac{ac}{c}-\frac{b}{c}=a-\frac{b}{c}$.

The terms of the preceding fractions were simple quantities, but if we had fractions, the terms of which were polynomials, we should have to perform, by the rules given for complex quantities, the operations indicated upon simple quantities; it is thus that we have

$$\frac{a^2+b^2}{c+d}\times\frac{a-b}{c-d}=\frac{(a^2+b^2)\,(a-b)}{(c+d)\,(c-d)}=\frac{a^3+ab^2-a^2b-b^3}{c^2-d^2}.$$

The quotient of the fraction

$$\frac{a^2+b^2}{c+d} \text{ divided by } \frac{a-b}{c-d},$$

is $\frac{a^2+b^2}{c+d}\times\frac{c-d}{a-b}=\frac{(a^2+b^2)\,(c-d)}{(c+d)\,(a-b)}=\frac{a^2c+b^2c-a^2d-b^2d}{ac+ad-bc-bd}$,

and so of other operations.

54. Understanding what precedes, we can resolve an equation of the first degree, however complicated.

If we have, for example, the equation

$$\frac{(a+b)\,(x-c)}{a-b}+4b=2x-\frac{ac}{3a+b},$$

we begin by making the denominators to disappear, indicating only the operations; it becomes then

$$(a+b)(x-c)(3a+b)+4b(a-b)(3a+b)=2x(a-b)(3a+b)-ac(a-b);$$

performing the multiplications we have

$$3a^2x+4abx+b^2x-3a^2c-4abc-b^2c+12a^2b-8ab^2-4b^3=$$
$$6a^2x-4abx-2b^2x-a^2c+abc;$$

transposing to one member all the terms involving x, it becomes

$$-3a^2x+8abx+3b^2x=2a^2c+5abc+b^2c-12a^2b+8ab^2+4b^3,$$

from which we deduce

$$x=\frac{2a^2c+5abc+b^2c-12a^2b+8ab^2+4b^3}{-3a^2+8ab+3b^2}.$$

Of questions having two unknown quantities, and of negative quantities.

55. The questions, which we have hitherto considered, involve only one unknown quantity, by means of which, with the known quantities, are expressed all the conditions of the question. It is often more convenient, in some questions, to employ two unknown quantities, but then there must be, either expressed or implied, two conditions, in order to form two equations, without which the two unknown quantities cannot be determined at the same time.

The question in art. 3, especially as it is enunciated in art. 4, presents itself naturally with two unknown quantities, that is, with both the numbers sought. Indeed if we denote

the least by x,
the greatest by y,
their sum by a,
their difference by b,

we have by the enunciation of the question,

$$x+y=a$$
$$y-x=b.$$

Each of these two quantities being considered by itself, we can determine one of the unknown quantities. If we take the second, for example, we deduce the value of y, which is

$$y=b+x,$$

a value which seems at first to teach us nothing with regard to what we are seeking, since it contains the quantity x, which is not given; but if instead of the unknown quantity y in the first equation, we put this, its equivalent; the equation, containing now only one unknown quantity x, will give the value of x by the process already taught.

We have in fact by this substitution

$$x+b+x=a,$$

or $$2x+b=a,$$

or lastly $$x=\frac{a-b}{2};$$

and putting this value of x in the expression for y,

$$y = b + x = b + \frac{a-b}{2} = \frac{a+b}{2};$$

we have then for the two unknown numbers the same expressions as in art. 3.

It is easy to see indeed, that the above solution does not differ essentially from that of art. 3; only I have supposed and resolved the second equation $y - x = b$, which I contented myself with enunciating in common language in the article cited; and from it I deduced, without algebraic calculation, that the greater number was $x + b$.

56. I take another question.

A labourer having worked for a person 12 days, and having with him, during the 7 first days, his wife and son, received 74 francs; he worked afterward with the same person 8 days more, during 5 of which, he had with him his wife and son, and he received at this time 50 francs; how much did he earn per day himself, and how much did his wife and son earn?

Let x be the daily wages of the man,
y that of his wife and son;
12 days' work of the man will amount to $12\,x$,
7 days' work of his wife and son . . $7\,y$;
we have then by the first statement of the question,

$$12\,x + 7\,y = 74;$$

8 days' work of the man will give $8\,x$,
and 5 days' work of his wife and son $5\,y$;
we have then by the second statement

$$8\,x + 5\,y = 50.$$

Proceeding as in the preceding question, we take the value of y in the first equation, which is

$$y = \frac{74 - 12\,x}{7},$$

and substitute this value in the second, multiplying it by 5, the coefficient, and it becomes

$$8\,x + \frac{370 - 60\,x}{7} = 50,$$

an equation, which contains only the unknown quantity x. By reducing it we have

$$56\,x + 370 - 60\,x = 350,$$
$$370 - 4\,x = 350;$$

transposing $-4x$ to the second member, and 350 to the first, we obtain

$$370 - 350 = 4x$$
$$20 = 4x$$
$$\tfrac{20}{4} = x$$
$$5 = x.$$

Knowing x, which we have just found equal to 5, if we place this value in the formula

$$y = \frac{74 - 12x}{7},$$

the second member will be determined, for we have

$$y = \frac{74 - 12 \times 5}{7} = \frac{74 - 60}{7} = \frac{14}{7} = 2;$$

thus $y = 2.$

The man then earned 5 francs per day, while his wife and son earned only 2.

57. The reader has perhaps observed, that in resolving the above equation $370 - 4x = 350$, I have transposed $4x$ to the second member. I have proceeded thus to avoid a slight difficulty, that would otherwise have occured, and, which I will now explain.

By leaving $4x$ in the first member, and transposing 370 to the second, we have

$$-4x = 350 - 370;$$

and reducing the second according to the rule in art. 19, there will result from it

$$-4x = -20.$$

But as we have avoided, in the preceding article, the sign $-$, which affects the quantity $4x$, by transposing this quantity to the other member; and as in like manner the quantity $350 - 370$ becomes by transposition $370 - 350$; and since a quantity, by being thus transferred from one member to the other changes the sign (10), it is evident that we may come to the same result by simply changing the sign of each of the quantities $-4x$, $+350 - 370$, which gives

$$4x = -350 + 370,$$

or $4x = 370 - 350$

which is the same as

$$370 - 350 = 4x.$$

We might also change the signs after reduction, and the equation

$$-4x = -20$$

becomes, as above,

$$4x = 20.$$

It follows from this, that *we may transpose indifferently to one member or to the other, all the terms involving the unknown quantity, observing merely to change the signs of the two members in the result, when the unknown quantity has the sign* —.

58. Having undertaken, by means of letters, a general solution of the problem of art. 56, I will now examine a particular case. I suppose that the first sum received by the labourer to be 46 francs, and the second 30, the other circumstances remaining as before; the equations of the question will then be

$$12x + 7y = 46,$$
$$8x + 5y = 30.$$

The first gives

$$y = \frac{46 - 12x}{7};$$

multiplying this value by 5, in order to substitute it in the place of $5y$, in the second, we have

$$8x + \frac{230 - 60x}{7} = 30;$$

the denominator being made to disappear, it becomes

$$56x + 230 - 60x = 210,$$

or
$$56x - 60x = 210 - 230$$

or
$$-4x = -20$$

and the signs being changed agreeably to what has just been remarked,

$$4x = 20,$$
$$x = \tfrac{20}{4} = 5.$$

If we substitute this value instead of x in the expression for y, it will become

$$y = \frac{46 - 60}{7}$$

or
$$y = \frac{-14}{7}.$$

Now how are we to interpret the sign —, which affects the insolated quantity 14? We understand its import, when there are two quantities separated from each other by the sign —, and

when the quantity to be subtracted is less than that from which it is to be taken; but how can we subtract a quantity when it is not connected with another in the member where it is found? To clear up this difficulty it is best to go back to the equations, which express the conditions of the question; for the nearer we approach to the enunciation, the closer shall we bring together the circumstances which have given rise to the present uncertainty.

I resume the equation

$$12x + 7y = 46,$$

I put in the place of x its value 5, and it becomes

$$60 + 7y = 46.$$

This equation by mere inspection presents an absurdity. It is impossible to make the number 46 by adding any thing to the number 60, which exceeds it already.

I take also the second equation

$$8x + 5y = 30,$$

and putting 5 in the place of x, I find

$$40 + 5y = 30;$$

the same absurdity as before, since the number 30 is to be formed by adding something to the number 40.

Now the quantities $12x$ or 60 in the first equation, $8x$ or 40 in the second, represent what the labourer earned by his own work; the quantities $7y$ and $5y$ stand for the earnings of his wife and son, while the numbers 46 and 30 express the sum given as the common wages of the three; we must see then at once in what consists the absurdity.

According to the question, the labourer earned more by himself than he did by the assistance of his wife and son; it is impossible then to consider what is allowed to the woman and son, as augmenting the pay of the labourer.

But if, instead of counting the allowance made to the two latter persons as positive, we regard it as a charge placed to the account of the labourer, then it would be necessary to deduct it from his wages; and the equations would no longer involve a contradiction, as they would become

$$60 - 7y = 46,$$
$$40 - 5y = 30;$$

we deduce from the one as well as from the other

$$y = 2;$$

and we conclude from it, that if the labourer earned 5 francs per day, his wife and son were the occasion of an expense of 2 francs, which may otherwise be proved thus.

For 12 days' labour he received

$$5 \times 12 \text{ or } 60 \text{ francs};$$

the expense of his wife and son for 7 days is

$$2 \times 7 \text{ or } 14 \text{ francs};$$

there remain then 46 francs.

For 8 days' labour he receives

$$5 \times 8 \text{ or } 40 \text{ francs};$$

the expense of his wife and son for 5 days is

$$2 \times 5 \text{ or } 10 \text{ francs},$$

there remain 30 francs.

It is very clear then, that in order to render the proposed problem with the first conditions possible, instead of the enunciation in article 56, we must substitute this;

A labourer worked for a person 12 *days, having had with him the* 7 *first days, his wife and son at a certain expense, and he received* 46 *francs; he worked afterwards* 8 *days, during* 5 *of which, he had with him his wife and son at expense as before, and he received* 30 *francs. It is required to find how much he earned per day, and what was the sum charged him per day on account of his wife and son.*

Calling x the daily wages of the labourer, and y the daily expense of his wife and son, the equations of the problem will evidently be

$$12x - 7y = 46,$$
$$8x - 5y = 30;$$

and being resolved after the manner of those in art. 56, they will give

$$x = 5 \text{ francs}, \quad y = 2 \text{ francs}.$$

59. In every case, where we find for the value of the unknown quantity, a number affected with the sign $-$, we can rectify the enunciation in a manner analogous to the preceding, by examining with care what that quantity is among those, which are additive in the first equation, which ought to be subtractive in the second; but algebra supercedes the use of every inquiry of this kind, when we have learnt to make a proper use of expressions affected with the sign $-$; for these expressions being deduced from the equations of the problem must *satisfy* those

equations; that is to say, by subjecting them to the operations indicated in the equation, we ought to find for the first member a value equal to that of the second. Thus the expression $\frac{-14}{7}$ drawn from the equations

$$12x + 7y = 46,$$
$$8x + 5y = 30,$$

must, consistently with the value of $x = 5$, as deduced from these same equations, verify them both.

The substitution of the value of x gives in the first place

$$60 + 7y = 46,$$
$$40 + 5y = 30.$$

It remains to make the substitution of $\frac{-14}{7}$ in the place of y; and for this purpose we must multiply by 7 and by 5, having regard to the sign $-$, with which the numerator of the fraction is affected.

If we apply the rule relative to the signs given in art. 42 for division, we have

$$\frac{-14}{7} = -2;$$

besides, by the rule for the signs in multiplication we find

$$7 \times -2 = -14,$$
$$5 \times -2 = -10.$$

Hence the equations

$$60 + 7y = 46 \quad \text{and} \quad 40 + 5y = 30,$$

become respectively

$$60 - 14 = 46 \quad \text{and} \quad 40 - 10 = 30,$$

and are verified, not by adding the two parts of the first member, but in reality by subtracting the second from the first, as was done above, after considering the proper import of the equations.

60. The problem in art. 58 does not admit of a solution in the sense in which it is first enunciated; that is to say, by addition, or regarding as an accession the sum considered with reference to the wife and son of the labourer; neither does the second enunciation consist with the data of the problem in art. 56.

If we were to consider in this case y, as expressing a deduction, the equations thus obtained

$$12x - 7y = 74,$$
$$8x - 5y = 50,$$

would give

$$x = 5 \quad \text{and} \quad y = \frac{-14}{7};$$

and the substitution of the value of x would immediately change the equations to

$$60 - 7y = 74,$$
$$40 - 5y = 50.$$

The absurdity of these results is precisely contrary to that of the results in art. 58, since it relates to remainders greater than the numbers 60 and 40, from which the quantities $7y$ and $5y$ are to be subtracted.

The sign minus which belongs to the expression of y, implies an absurdity; but this is not all, it does it away also; for according to the rule for the signs,

$$\frac{-14}{7} = -2$$

and

$$-7 \times -2 = +14$$
$$-5 \times -2 = +10.$$

Thus the equations

$$60 - 7y = 74, \qquad 40 - 5y = 50,$$

become

$$60 + 14 = 74, \qquad 40 + 10 = 50,$$

and are verified by addition; consequently the quantities $-7y$ and $-5y$, transformed into $+14$, $+10$, instead of expressing expenses incurred by the labourer, are regarded as a real gain. We are brought back then in this case also to the true enunciation of the question.

61. We perceive by the preceding examples, that *there may be in the enunciations of a problem of the first degree, certain contradictions, which algebra not only makes known, but points out also, how they may be reconciled, by rendering subtractive certain quantities which had been regarded as additive, or additive certain quantities which had been regarded as subtractive, or by giving to the unknown quantities values affected with the sign* —.

See then what is to be understood, when we speak of values affected by the sign —, and of what are called *negative solutions* resolving, in a sense opposite to the enunciation, the question in which they occur.

It follows from this, that we may regard, as but one single question, those, the enunciations of which are connected together in such a manner, that the solutions, which satisfy one of the enunciations, will, by a mere change of sign, satisfy the other also.

62. Since negative quantities resolve in a certain sense the problems, which give rise to them, it is proper to inquire a little more particularly into the use of these quantities, and to settle once for all the manner of performing operations in which they are concerned.

We have already made use of the rule for the signs, which had been previously determined for each of the fundamental operations; but the rules have not been demonstrated with reference to insolated quantities. In the case of subtraction, for example, we supposed that there was to be taken from a the expression $b - c$, in which the negative quantity c was preceded by a positive quantity b. Strictly speaking, the reasoning does not depend upon the value of b; it would still apply when $b = 0$, which reduces the expression $b - c$ to $-c$. But the theory of negative quantities being at the same time one of the most important and most difficult in algebra, it should be established upon a sure basis. To effect this, it is necessary to go back to the origin of negative quantities.

The greatest subtraction, that can be made from a quantity, is to take away the quantity itself, and in this case we have zero for a remainder; thus $a - a = 0$. But when the quantity to be subtracted exceeds that from which it is to be taken, we cannot subtract it entirely; we can only make a reduction of the quantity to be subtracted equal to the quantity from which it was to be taken. When, for example, it is required to subtract 5 from 3, or when we have the quantity $3 - 5$; to take in the first place 3 from 5, we decompose 5 into two parts 3 and 2, the successive subtraction of which will amount to that of 5, and thus, instead of $3 - 5$, we have the equivalent expression $3 - 3 - 2$, which is reduced to -2. The sign $-$, which precedes 2, shows what is necessary to complete the subtraction; so that, if we had added 2 to the first of the quantities, we should have had $3 + 2 - 5$, or zero. We express then with the help of algebraic signs, the idea that is to be attached to a negative quantity $-a$, by forming the equation $a - a = 0$, or by regarding the symbols $a - a$, $b - b$, &c. as equivalent to zero.

This being supposed, it will be understood, that if we add to any quantity whatever the symbol $b - b$, which in reality is only zero, we do not change the value of this quantity, and that consequently the expression $a + b - b$ is nothing else but a different manner of writing the quantity a which is also evident from the consideration, that $+ b$ and $- b$ destroy each other.

But having by this change of form introduced $+ b$ and $- b$ into the same expression with a, we see that in order to subtract any one of these quantities, it is sufficient to efface it. If it were $+ b$ that we would subtract, we efface it, and there remains $a - b$, which accords with the rule laid down in art. 2; if on the other hand it were $- b$, we efface this quantity and there would remain $a + b$, as might be inferred from art. 20.

With respect to multiplication it will be observed, that the product of $a - a$ by $+ b$ must be $ab - ab$, because the multiplicand being equal to zero, the product must be zero; and the first term being ab, the second must necessarily be $- ab$ to destroy the first.

We infer from this, that $- a$, multiplied by $+ a$, must give $- ab$.

By multiplying a by $b - b$, we have still $ab - ab$, because the multiplier being equal to zero, the product will also be equal to zero; it is therefore necessary that the second term should be $- ab$ to destroy the first $+ ab$.

Whence $+ a$ multiplied by $- b$ must give $- ab$.

Lastly, if we multiply $- a$ by $b - b$, the first term of the product being, according to what has just been proved, $- ab$, it is necessary that the second term should be $+ ab$, as the product must be nothing when the multiplier is nothing.

Whence $- a$, multiplied by $- b$ gives $+ ab$.

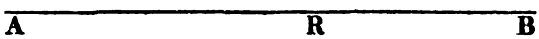
By collecting these results together we may deduce from them the same rules as those in art. 31 (A).

As the sign of the quotient, combined with that of the divisor according to the rules proper for multiplication, must produce the sign of the dividend, we infer from what has just been said, that the rule for the signs given in art. 42, corresponds with that which it is necessary to observe in fact, and that consequently, *simple quantities, when they are insolated, are combined with respect to their signs, in the same manner, as when they make a part of polynomials.*

63. According to these remarks we may always, when we meet with negative values, go back to the true enunciation of the question resolved, by seeking in what manner these values will satisfy the equations of the proposed problem; this will be confirmed by the following example, which relates to numbers of a different kind from those of the question in art. 56.

64. *Two couriers set out to meet each other at the same time from two cities, the distance of which is given; we know how many miles*(a) *each travels per hour, and we inquire at what point of the route between the two cities they will meet.*

To render the circumstances of the question more evident, I have subjoined a figure in which the points A and B represent the places of departure of the couriers.



I denote the things given, and those required in the usual way, by small letters.

a the distance in miles of the points of departure A and B,

b the number of miles per hour, which the courier from A travels,

c the number of miles per hour which the courier from B travels.

The letter R being placed at the point of meeting of the two couriers, I shall call x the distance AR passed over by the first, y the distance BR passed over by the second, and as

$$AR + BR = AB,$$

I have the equation

$$x + y = a.$$

Considering that the distances x and y are passed over in the same time, we remark that the first courier, who travels a number b of miles in an hour, will employ, in passing over the distance x, a time denoted by $\frac{x}{b}$.

Also the second courier, who travels c miles in an hour, will employ, in passing over the distance y, a time denoted by $\frac{y}{c}$; we have then

$$\frac{x}{b} = \frac{y}{c}.$$

(a) In the original the distance is given in *kilometres*. It is here *expressed by miles to avoid perplexing the learner.*

The equations of the question therefore will be

$$x+y=a$$
$$\frac{x}{b}=\frac{y}{c}.$$

Making the denominator b of the second to disappear, we have

$$x=\frac{by}{c};$$

putting this value in the place of x in the first equation, it becomes

$$\frac{by}{c}+y=a,$$

and we deduce from it

$$by+cy=ac, \quad \text{whence} \quad y=\frac{ac}{b+c}.$$

Substituting this value of y in the expression for the value of x, we obtain

$$x=\frac{b}{c}\times\frac{ac}{b+c} \quad \text{or} \quad x=\frac{abc}{c(b+c)} \text{ (51)},$$

or lastly

$$x=\frac{ab}{b+c} \text{ (38)}.$$

As the sign $-$ does not enter into the values of x and y, it is evident that whatever numbers are put for the letters a, b, c, we shall always find x and y with the sign $+$, and therefore the question proposed will be resolved in the precise sense of the enunciation. Indeed it is readily perceived, that in every case where two persons set off from different points and travel toward each other they must necessarily meet.

65. I will now suppose, that the two couriers proceed in the same direction, and that the one, who sets out from A, is pursuing the one who sets out from B, and who is travelling toward the same point C, placed beyond B, with respect to A.

A B R C

It is evident that in this case, the courier, who starts from the point A, cannot come up with the courier who sets off from the point B, except he travels faster than this last, and the point of coming together R cannot be between A and B, but must be beyond B, with respect to A.

Having the same things given as before, and observing that when

$$AR-BR=AB,$$

we have

$$x - y = a.$$

The second equation

$$\frac{x}{b} = \frac{y}{c}$$

expressing only the equality of the times employed by the couriers in passing over the distances *A R* and *B R*, undergoes no change.

The above equations, being resolved like the former ones, give

$$x = \frac{by}{c},$$

$$\frac{by}{c} - y = a, \qquad by - cy = ac,$$

$$y = \frac{ac}{b-c},$$

$$x = \frac{b}{c} \times \frac{ac}{b-c} = \frac{abc}{c(b-c)},$$

and lastly

$$x = \frac{ab}{b-c}.$$

Here the values of x and y will not be positive, except when b is taken greater than c, that is to say, except the courier starting from the point *A* be supposed to travel faster thán the other.

If, for example, we make

$$b = 20, \qquad c = 10,$$

we have

$$x = \frac{20a}{20-10} = \frac{20a}{10} = 2a,$$

$$y = \frac{10a}{20-10} = \frac{10a}{10} = a;$$

from which it follows, that the point of their coming together is distant from the point *A* twice *A B*.

If we now suppose b smaller than c, and take, for example

$$b = 10, \qquad c = 20,$$

we find

$$x = \frac{10a}{10-20} = \frac{10a}{-10} = -a,$$

$$y = \frac{20a}{10-20} = \frac{20a}{-10} = -2a.$$

These values being affected with the sign $-$, make it evident, that the question cannot be resolved in the sense in which it is enunciated; and indeed it is absurd to suppose that the courier

setting out from the point *A*, and proceeding only 10 miles in an hour, should ever be able to overtake the courier setting out from the point *B* and travelling 20 miles per hour, and who is in advance of the first.

66. Nevertheless, these same values resolve the question in a certain sense; for by substituting them in the equations

$$x - y = a$$

$$\frac{x}{b} = \frac{y}{c},$$

we have by the rule for the signs

$$-a + 2a = a$$

$$-\frac{a}{10} = -\frac{2a}{20},$$

equations which are satisfied; since by making the reductions, the first member becomes equal to the second; and if we attend to the signs of the terms, which compose the first, we shall see how it is necessary to modify the enunciation of the question, in order to do away the absurdity.

Indeed, it is the distance a corresponding to x, and passed over by the first courier, which is in reality subtracted from the distance $2a$, corresponding to y and passed over by the second courier; it is then just as if we had changed y into x, and x into y, and had supposed that the courier starting from the point *B* had run after the other.

This change in the enunciation, produces also a change in the direction of the routes of the couriers; they are no longer travelling toward the point *C*, but in an opposite manner toward the point *C'*, as represented in the figure below;

C' R' A B R C

and their coming together takes place in *R'*. The result from this is

$$BR' - AR' = AB,$$

which gives

$$y - x = a;$$

we have besides constantly

$$\frac{x}{b} = \frac{y}{c},$$

and we find

$$x = \frac{ab}{c-b} = \frac{10a}{20-10} = a,$$

$$y=\frac{a\,c}{c-b}=\frac{20\,a}{20-10}=2\,a,$$

positive values, which resolve the question in the precise sense in which it is enunciated.

67. The question we have been considering presents a case, in which it is in every sense absurd. This occurs when we suppose the two couriers to travel equally fast. It is evident that in whatever direction we suppose them to move, they can never come together, since they preserve constantly the interval of their points of departure. This absurdity, which no modification in the enunciation can remove, is very conspicuous in the equations.

We have now $b=c$, since the couriers travelling equally fast pass over the same space in an hour; the equation

$$\frac{x}{b}=\frac{y}{c}$$

becomes

$$\frac{x}{b}=\frac{y}{b}$$

and gives

$$x=y.$$

Thus the equation

$$x-y=a$$

reduces itself to

$$x-x=a \quad \text{or} \quad 0=a,$$

a result sufficiently absurd, since it supposes a quantity a, the magnitude of which is given, to be nothing.

68. This absurdity shews itself in a manner very singular in the values of the unknown quantities

$$x=\frac{a\,b}{b-c}, \qquad y=\frac{a\,c}{b-c};$$

their denominator becoming 0 when $b=c$ we have

$$x=\frac{a\,b}{0}, \qquad y=\frac{a\,c}{0}.$$

We do not easily perceive what may be the quotient of a division when the divisor is zero; we see merely that if we consider b as nearly equal to c, the values of x and y become very great. To be convinced of this, we need only take

$$b=6 \text{ miles}, \qquad c=5{,}8 \text{ miles}$$

we then have

$$x=\frac{6\,a}{0{,}2}=30\,a$$

$$y=\frac{5{,}8\,a}{0{,}2}=29\,a.$$

If further we take $b=6$, $c=5{,}9$

we have

$$x = \frac{6a}{0,1} = 60\,a,$$

$$y = \frac{5,9\,a}{0,1} = 59\,a.$$

If moreover we make

$$b = 6, \qquad c = 5,99,$$

it becomes

$$x = \frac{6a}{0,01} = 600\,a,$$

$$y = \frac{5,99\,a}{0,01} = 599\,a,$$

and it is manifest, that as the divisor diminishes in proportion to the smallness of the assumed difference of the numbers b and c, we obtain values more and more increased in magnitude.

But as a quantity however minute can never be taken for zero, it follows that however small we make the difference of the numbers represented by the letters b and c, and however great may be the consequent values of x and y, we never attain to those which answer to the case where $b = c$.

Since these last cannot be represented by any number, however great we suppose it, they are said to be *infinite;* and every expression of the form $\frac{m}{0}$, the denominator of which is zero, is regarded as the symbol of *infinity.*

This example shows that mathematical *infinity* is a negative idea, since we at length get it only by the impossibility of assigning a quantity that can resolve the question.

We may ask here how the values

$$x = \frac{ab}{0}, \qquad y = \frac{ac}{0}$$

satisfy the equations proposed; for it is an essential characteristic of algebra, that the symbols of the values of unknown quantities, whatever they may be, being subjected to the operations indicated upon these quantities, shall satisfy the equations of the problem.

By substituting them in the equations

$$x - y = a,$$

$$\frac{x}{b} = \frac{y}{b},$$

which answer to the case where $b = c$, we have by the first,

$$\frac{ab}{0}-\frac{ab}{0}=a,$$

or $$\frac{ab-ab}{0}=a, \quad \text{or} \quad ab-ab=a\times 0,$$

or lastly $$0=0, \quad \text{since} \quad a\times 0=0.$$

The second equation gives, under the same condition,

$$\frac{ab}{0\times b}=\frac{ab}{0\times b};$$

the two members of each equation becoming equal, the equations are satisfied.

It remains still to explain how the notion indicated by the expression $\frac{ab}{0}$, removes the absurdity of the result found in art. 57. For this purpose, let the two members of the equation

$$x-y=a$$

be divided by x, which gives

$$1-\frac{y}{x}=\frac{a}{x};$$

and as the equation

$$\frac{x}{b}=\frac{y}{b}$$

gives $x=y$, the first becomes

$$1-1=\frac{a}{x}, \quad \text{or} \quad 0=\frac{a}{x}.$$

The error here consists in the quantity $\frac{a}{x}$, by which the second member exceeds the first; but this error becomes smaller and smaller, in proportion to the assumed magnitude of x. It is then with reason, that algebra gives for x an expression, which cannot be represented by any number, however great, but which, as it proceeds in the order of numbers becoming greater and greater, points out in what manner we may reduce more and more the error of the supposition.

69. If the couriers travelling equally fast, and in the same direction, had set out from the same point, their coming together could not be said to take place at any particular point, since they would be together through the whole extent of their route. It may be worth while to see how this circumstance is represented by the values, which the unknown quantities x and y assume in this case.

The points A and B being coincident, we have on this supposition $a=0$, and constantly $b=c$; it follows then, that

$$x=\frac{0.b}{0}=\frac{0}{0}, \quad y=\frac{0.c}{0}=\frac{0}{0}.$$

In order to interpret these values, that indicate a division, in which the dividend and divisor are each nothing, I go back to the equations of the question. The first becoming

$$x-y=0 \quad \text{gives} \quad x=y;$$

and substituting this value in the second equation, which is

$$\frac{x}{b}=\frac{y}{b}, \quad \text{it becomes} \quad \frac{y}{b}=\frac{y}{b}.$$

The last equation having its two members *identical*, that is to say, composed of the same terms with the same sign is verified, whatever value is assigned to y, and this unknown quantity can never be determined. Besides, it is evident that the equation

$$\frac{x}{b}=\frac{y}{b} \quad \text{becomes} \quad x=y,$$

and consequently can express nothing more than the first.* The only result both from the one, and from the other is, that the two couriers are always together, since the distances x and y from the point A are equal, their value in other respects remains indeterminate. The expression $\frac{0}{0}$ then is here a symbol of an indeterminate quantity. I say here, for there are cases where it is not; but the expression has not then the same origin as the preceding.

70. To give an example, let there be

$$\frac{a\,(a^2-b^2)}{b\,(a-b)}.$$

This quantity becomes $\frac{0}{0}$ in its present form when $a=b$; but if we reduce it first to its most simple expression, by suppressing the factor $a-b$, common to the numerator and denominator, we find

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* For the sake of conciseness, analysts apply to the same equations the epithet, *identical.*

$\frac{y}{b}=\frac{y}{b}$ is an identical equation, $5-3x=5-3x$ is another, and when two equations express only the same thing, we say that these equations also are identical.

$$\frac{a(a+b)}{b},$$

which gives $2a$ when $a=b$.

It is not the same with the values of x and y, found in the preceding article, for they are not susceptible of being reduced to a more simple expression.

It follows, from what I have just said, that when we meet with an expression which becomes $\frac{0}{0}$, it is proper before pronouncing upon its value, to see if the numerator and denominator have not a common factor, which becoming nothing, renders the two terms at the same time equal to zero, and which being suppressed, the true value of the proposed expression is obtained. There are, notwithstanding, some cases which elude this method, but the limits of this work will only allow me to note the *analytical fact.* It belongs properly to the differential calculus to give the general processes for finding the true value of quantities, which become $\frac{0}{0}$.

71. It is very evident, from what has been said, that *algebraic solutions either answer perfectly to the conditions of a problem, when it is possible, or they indicate a modification to be made in the enunciation, when the things given imply contradictions that cannot be reconciled; or lastly, they make known an absolute impossibility, when there is no method of resolving with the same things given, a question analogous in a particular sense to the one proposed.*

72. It may be remarked, that in the different solutions of the preceding question, the changing of the signs of the unknown quanties x and y corresponds to a change in the direction of the journeys represented by the unknown quantities. When the unknown quantity y was counted from B towards A, it had in the equation

$$x+y=a,$$

the sign $+$, and it takes the sign $-$ for the second case, when the motion is in the opposite direction from B towards C, art. 65, since we had for the first equation

$$x-y=a.$$

By changing the sign in the second equation

$$\frac{x}{b}=\frac{y}{c},$$

we have

$$\frac{x}{b}=\frac{-y}{c},$$

a result which does not differ from that given in the article cited; but it should be observed, that the journey y being made up of multiples of the space c passed over in an hour by the courier from B, and this space having the same direction as the space y, ought to be supposed to have the same sign, and consequently to take the sign $-$, when $-$ is applied to y; we have accordingly

$$\frac{x}{b}+\frac{-y}{-c}, \quad \text{or} \quad \frac{x}{b}+\frac{y}{c}.$$

A simple change of sign then is sufficient to comprehend the second case of the question in the first, and it is thus that algebra gives at the same time the solution of several analogous questions.

We have a striking example of this in the problem of art. 15. It is here supposed that the father owed the son a sum d; if we would resolve the question on the contrary hypothesis, that is, by supposing that the son owed the father the sum d, it would be sufficient to change the sign of d in the value of x, and we have

$$x=\frac{bc-d}{a+b}.$$

If we suppose neither to owe the other any thing, we must make $d=0$, and then the equation would be

$$x=\frac{bc}{a+b}.$$

Nothing can be easier than to verify the two solutions by putting anew the problem into an equation for each of the cases, which we have enunciated.

73. It was only to preserve an analogy between the problems 56 and 64, that I have employed two unknown quantities in the second. Each may be resolved with only one unknown quantity; for when we say that the labourer received 74 francs for 12 days' work performed by himself and 7 days' work by his wife and son, it follows that, if we call y the daily wages of the woman and son, and take $7y$ from 74 francs, there will remain $74-7y$ for the 12 days' labour of the man; from which we infer that he earned $\frac{74-7y}{12}$ per day.

By a similar calculation for the 8 days' service, we find that he earned $\frac{50-5y}{8}$ per day.

Putting the two quantities equal to each other we form the equation

$$\frac{74-7y}{12}=\frac{50-5y}{8}.$$

Also in the question of art. 64,

A R B

if x represent the course AR of the courier from A, $BR=a-x$ would be that of the courier who set off from B towards A. These two distances being passed over in the same time by the couriers whose rate of travelling per hour in miles is denoted by the numbers b and c respectively, we have

$$\frac{x}{b}=\frac{a-x}{c},$$

whence

$$cx=ab-bx,$$

$$x=\frac{ab}{b+c}.$$

The difference between the solutions, which I have now given and those of articles 56 and 64, consists merely in this, that we have formed and resolved the first equation by the assistance of ordinary language, without employing algebraic characters, and it is manifest, that the further we carry this, the less will remain to be effected by the other.

74. We sometimes add to the problem of art. 64 a circumstance, which does not render it more difficult.

A R C B

We suppose that the courier, who starts from B, *sets off a number* d *of hours before the other, who goes from* A.

It is evident that this amounts only to a change of the point of departure of the first, for if he travelled a number c of miles per hour, he would pass over the space $BC=cd$ in d hours, and would be at the point C, when the other courier set off from A; so that the interval of the points of departure would be

$$AC=AB-BC=a-cd.$$

By writing then $a-cd$ in the place of a in the equation of the preceding article, we have

$$\frac{x}{b}=\frac{a-cd-x}{c}$$

$$x=\frac{ab-bcd}{b+c}.$$

If the couriers proceeded in the same direction, the interval of

A B C R

the points of departure would be

$$AC = AB + BC = a + cd;$$

and the distance passed over by the courier from the point A would be AR, while that passed over by the other courier would be

$$CR = AR - AC;$$

we have then

$$\frac{x}{b} = \frac{x - a - cd}{c},$$

whence

$$x = \frac{ab + bcd}{b - c}.$$

75. Enunciated in this manner the problem presents a case, in which the interpretation of the negative value found for x is attended with some difficulty; it is when the couriers being supposed to proceed in opposite directions, we give to the number d a value such, that the space BC represented by cd becomes greater than a, which represents AB.

C R A B

Now the courier from the point B arrives at C on the other side of A at the moment, when the courier from A sets off towards B; there is then an absurdity in supposing that the two couriers can thus come together.

If we should take, for example,

$$a = 400^{\text{mls.}}, \quad b = 12^{\text{mls.}}, \quad c = 8^{\text{mls.}}, \quad d = 60^{\text{h}'},$$

there would result from it $cd = 480^{\text{mls.}}$, thus the point C would be $80^{\text{mls.}}$ on the other side of A, with respect to the point B; but we find,

$$x = \frac{400 \,.\, 12 - 60 \,.\, 8 \,.\, 12}{8 + 12} = \frac{400 \,.\, 3 - 60 \,.\, 2 \,.\, 12}{2 + 3}$$

$$= \frac{1200 - 1440}{5} = -\frac{240}{5} = -48.$$

Thus the coming together of the couriers takes place in a point R, $48^{\text{mls.}}$ on the other side of the point A, but between A and C; although it seems that the courier from B, being supposed to continue his journey beyond the point C, can be overtaken by the other courier only after he has passed this point.

To understand the question resolved in this sense, we may substitute in the place of x the negative member $-m$, and the

equation becomes

$$-\frac{m}{b}=\frac{a-cd+m}{c},$$

or by changing the signs in the two members,

$$\frac{m}{b}=\frac{cd-a-m}{c}.$$

We see that the distance passed over by the courier from the

C R A B

point B, is $cd-a-m$, or what remains of BC after AB and AR are subtracted, that is CR, and that $AC=cd-a$. This is just what would take place if the second courier had started immediately from the point C, where he is at the departure of the first; but as they travel in opposite directions, they must necessarily meet between A and C. Thus, this case is similar to the first of those of art. 74, where it is sufficient to change $a-cd$ into $cd-a$, in order to obtain the value, which m has according to the above equation.*

76. The problem of art. 56, taken in its most enlarged sense, may be enunciated as follows;

A labourer having passed a number a *of days in a family, and having with him his wife and son during a number* b *of days, received a sum* c*; he lived afterward in the same family a number* d *of days; he had with him this time his wife and son during a number* e *of days, and he received a sum* f*; we inquire what he earned per day, and what was allowed per day to his wife and son.*

Let x represent constantly the daily wages of the labourer, and y that of his wife and son; for the number a of days he has ax, and for the number b of days his wife and son have by, so that,

$$ax+by=c;$$

for the number d of days, he has dx, and for the number e of days his wife and son have ey, thus,

$$dx+ey=f.$$

These are the general equations of the question.

We deduce from the first

$$x=\frac{c-by}{a};$$

multiplying this value by d, in order to substitute it in the place

* See note at the end of the Elements of Algebra.

of x in the second equation, we have

$$dx = \frac{cd - bdy}{a},$$

and consequently

$$\frac{cd - bdy}{a} + ey = f.$$

By making the denominator to disappear, we obtain

$$cd - bdy + aey = af,$$

whence

$$aey - bdy = af - cd,$$

$$y = \frac{af - cd}{ae - bd}.$$

Having the value of y, if we substitute it instead of y expression for x, this last will be known

$$x = \frac{c - b\,\dfrac{af - cd}{ae - bd}}{a}.$$

To simplify this expression, we should, in the first place, p the multiplication indicated upon the quantities

$$b \quad \text{and} \quad \frac{af - cd}{ae - bd}, \qquad (51)$$

which gives

$$x = \frac{c - \dfrac{abf - bcd}{ae - bd}}{a};$$

and then reduce c to a fraction having the same denomin the fraction which accompanies it, and perform the subt of this fraction (53); and it becomes

$$x = \frac{\dfrac{ace - bcd - abf + bcd}{ae - bd}}{a},$$

or by being reduced

$$x = \frac{\dfrac{ace - abf}{ae - bd}}{a}.*$$

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* There might be some doubt as to the meaning of this e sion; but it is obviated by attending to the bar denoting di which is placed in the middle of the line. Thus, in the exp $x = \frac{A}{B}$, A represents the dividend, whether integral or fraction B the divisor, which may also be a whole number or a fraction.

Dividing by a (51) we have

$$x=\frac{a\,c\,e-a\,b\,f}{a^2\,e-a\,b\,d}.$$

Suppressing the factor a, common to the numerator and denominator (38), we find

$$x=\frac{c\,e-b\,f}{a\,e-b\,d}.$$

The values

$$x=\frac{c\,e-b\,f}{a\,e-b\,d},\quad y=\frac{a\,f-c\,d}{a\,e-b\,d},$$

are applied in the same manner as those, which we before found for literal equations, with only one unknown quantity; we substitute in the place of the letters, the particular numbers in the example selected.

We shall obtain the results in art. 56, by making

$$a=12,\quad b=7,\quad c=74,$$
$$d=8,\quad e=5,\quad f=50,$$

and those of art. 58, by making

$$a=12,\quad b=7,\quad c=46,$$
$$d=8,\quad e=5,\quad f=30.$$

77. The values of x and y are adapted not only to the proposed question; they extend also to all those, which lead to two equations of the first degree with two unknown quantities, since it is evident, that these equations are necessarily comprehended in the formulas,

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the expression $x=\frac{\frac{A}{C}}{B}$ signifies, that x is equal to the quotient of the fraction $\frac{A}{C}$ divided by B, and the expression $x=\frac{A}{\frac{B}{C}}$ indicates for x the quotient arising from A divided by the fraction $\frac{B}{C}$; and lastly, we denote by the expression $x=\frac{\frac{A}{C}}{\frac{B}{D}}$ the quotient resulting from the division of the fraction $\frac{A}{C}$ by the fraction $\frac{B}{D}$.

It will be perceived by these remarks, that it is necessary to place the bars according to the result, which we propose to express.

$$a x + b y = c,$$
$$d x + e y = f,$$

provided the letters a, b, d, e, denote the whole of the given quantities, by which the unknown quantities x and y are respectively multiplied, and the letters c and f the whole of the known terms, transposed to the second member.

Of the resolution of any given number of equations of the first degree, containing an equal number of unknown quantities.

[illegible] an equation has as many distinct conditions, as it [illegible] f these conditions furnishes [illegible] n, [illegible] s, that the unknown quantities are involved w[illegible] e have seen already in the problems with two [illegible] es; but if these unknown quantities are only [illegible] e, according to the method adopted in the pre[illegible] *take in one of the equations the value of one of the* [illegible] *ities, as if all the rest were known, and substitute this va*[illegible] *he other equations, which will then contain only the other* [illegible] *uantities.*

This operation, by which [illegible] minate one of the unknown quantities, is called *elimination.* [illegible] his way, if we have three equations with three unknown quantities, we deduce from them two equations with only two unknown quantities, which are to be treated as above; and having obtained the values of the two last unknown quantities, we substitute them in the expression for the value of the first unknown quantity.

If we have four equations with four unknown quantities, we deduce from them, in the first place, three equations with three unknown quantities, which are to be treated in the manner just described; having found the value of the three unknown quantities, we substitute them in the expression for the value of the first, and so on.

See an example of a question, which contains three unknown quantities and three equations.

79. *A person buys separately three loads of grain; the first, which contained* 30 *measures of rye,* 20 *of barley, and* 10 *of wheat, cost* 230 *francs;*

The second, which contained 15 *measures of rye,* 6 *of barley, and* 12 *of wheat, cost* 138 *francs;*

The third, which contained 10 *measures of rye,* 5 *of barley, and* 4 *of wheat, cost* 75 *francs;*

It is asked, what the rye, barley, and wheat cost each per measure?

Let x be the price of a measure of rye,

y that of a measure of barley,

z that of a measure of wheat.

To fulfil the first condition we observe, that

30 measures of rye are worth $30\,x$,

20 measures of barley are worth $20\,y$,

10 measures of wheat are worth $10\,z$;

and as the whole must make 230 francs, we have the equation

$$30\,x + 20\,y + 10\,z = 230.$$

For the second condition we have

15 measures of rye worth $15\,x$,

6 barley $6\,y$,

12 wheat $12\,z$,

and consequently

$$15\,x + 6\,y + 12\,z = 138.$$

For the third condition we have

10 measures of rye worth $10\,x$,

5 barley $5\,y$,

4 wheat $4\,z$,

and consequently

$$10\,x + 5\,y + 4\,z = 75.$$

The proposed question then will be brought into three equations;

$$30\,x + 20\,y + 10\,z = 230,$$
$$15\,x + 6\,y + 12\,z = 138,$$
$$10\,x + 5\,y + 4\,z = 75.$$

Before proceeding to the resolution, I examine the equations, to see if it is not possible to simplify them by dividing the two members of some one of them by the same number (12), and I find that the two members of the first may be divided by 10, and those of the second by 3. Having performed these divisions I have only to occupy myself with the equations

$$3\,x + 2\,y + z = 23,$$
$$5\,x + 2\,y + 4\,z = 46,$$
$$10\,x + 5\,y + 4\,z = 75.$$

As I can select any one of the unknown quantities in order to deduce its value, I take that of z in the first equation, because this unknown quantity having no coefficient, its value will be entire or without a divisor, as follows.

$$z = 23 - 3x - 2y.$$

This value being substituted for z in the second and third equations, they become

$$5x + 2y + 92 - 12x - 8y = 46,$$
$$10x + 5y + 92 - 12x - 8y = 75;$$

and reducing the first member of each, we find

$$92 - 7x - 6y = 46,$$
$$92 - 2x - 3y = 75.$$

To proceed with these equations, which contain only two unknown quantities, I take in the first the value of the unknown quantity y, and I obtain

$$y = \frac{92 - 46 - 7x}{6}, \quad \text{or} \quad y = \frac{46 - 7x}{6},$$

and by substituting this value in the second equation, it becomes

$$92 - 2x - 3 \times \frac{46 - 7x}{6} = 75.$$

The denominator 6 may be made to disappear by the usual method, but observing that the denominator is divisible by 3, I can simplify the fraction by multiplying it by 3, agreeably to article 54 of Arithmetic. I have then

$$92 - 2x - \frac{46 - 7x}{2} = 75.$$

The denominator 2 being made to disappear, it becomes

$$184 - 4x - 46 + 7x = 150;$$

the first member being reduced gives

$$138 + 3x = 150,$$

whence

$$x = \frac{150 - 138}{3} = \frac{12}{3}, \quad \text{or} \quad x = 4.$$

Substituting this value in the expression for that of y, I find

$$y = \frac{46 - 7 \times 4}{6} = \frac{46 - 28}{6} = \frac{18}{6}, \quad \text{or} \quad y = 3;$$

and by substituting these values in the expression for that of z we obtain

$$z = 23 - 3 \times 4 - 2 \times 3 = 23 - 12 - 6, \text{ or } z = 5.$$

It appears then, that the price of the rye per measure was 4 fr.
that of the barley 3,
that of the wheat 5.

This example, while it illustrates the method given in the preceding article, ought to be attended to on account of the abbreviations of calculation, which are performed in it.

80. I proceed now to resolve the following problem.

A man, who undertook to transport some porcelain vases of three different sizes, contracted that he would pay as much for each vessel that he broke, as he received for those, which he delivered safe.

He had committed to him two small vases, four of a middle size, and nine large ones; he broke the middle sized ones, delivered all the others safe, and received the sum of 28 francs.

There were afterwards committed to him seven small vases, three of the middle size, and five large ones; he rendered this time the small and the middle sized ones, but broke the five large ones, and he received only 3 francs.

Lastly he took charge of nine small vases, ten middle sized ones, and eleven large ones; all these last he broke, and received in consequence only 4 francs.

It is asked what was paid him for carrying a vase of each size.

Let x be the sum paid for carrying a small vase,
y that for carrying a middle sized one,
z that for carrying a large one.

It is evident that each sum, which the porter received, is the difference between what was due to him for the vessels delivered safe, and what he had to pay for those which were broken; accordingly the three conditions of the problem furnish respectively the following equations;

$$\begin{aligned} 2x - 4y + 9z &= 28, \\ 7x + 3y - 5z &= 3, \\ 9x + 10y - 11z &= 4. \end{aligned}$$

The first of these equations gives

$$x = \frac{28 + 4y - 9z}{2};$$

and by substituting this value, the second and third equations become

$$\frac{196 + 28y - 63z}{2} + 3y - 5z = 3,$$

$$\frac{252 + 36y - 81z}{2} + 10y - 11z = 4.$$

Making the denominators to disappear, we have

$$\begin{aligned} 196 + 28y - 63z + 6y - 10z &= 6, \\ 252 + 36y - 81z + 20y - 22z &= 8; \end{aligned}$$

reducing the first member of each, we obtain

$$196 + 34y - 73z = 6,$$
$$252 + 56y - 103z = 8;$$

t... lue of y in the first of these equations, we find

$$y = \frac{73z - 190}{34}.$$

By m... this value, the second equation becomes

$$252 + 56 \times \frac{73z - 190}{34} - 103z = 8;$$

b... l of the denominator 34, it is changed into

$$\ldots + 56 \times 73z - 56 \times 190 - 34 \times 103z = 34 \times 8$$

or i...

$$8568 + \ldots 40 - 3502z = 272.$$

The reduction of th... of this result gives

$$\ldots 2 = 272,$$

whence we deduce

$$\ldots \quad \text{or} \quad z = 4.$$

By going back ... f z to that of y, we have

$$y = \frac{73 \times 4 \ldots 190}{34} = \frac{102}{34}, \quad \text{or} \quad y = 3;$$

and with these two values, we ... l

$$x = \frac{28 + 4 \times 3 - 9 \times 4}{2} = \frac{28 + 12 - 36}{2} = \frac{4}{2}, \quad \text{or} \quad x = 2$$

The prices then were 2 fr. for carrying a small vase,
3 one of a middle siz
4 a large one.

This example is sufficient to show how to proceed in all s ilar cases.

81. It sometimes happens, that all the unknown quantitie not enter at the same time into all the equations; the met however, is not changed by this circumstance; it is suffici carefully to examine the connexion of the unknown quant in order to pass from one to the others.

Let there be, for example, the four equations

$$3u - 2y = 2,$$
$$2x + 3y = 39,$$
$$5x - 7z = 11,$$
$$4y + 3z = 41,$$

containing the unknown quantities u, x, y and z.

With a little attention we see, that by taking the value o

in the second equation, and substituting it in the third, the result containing only y and z, will, by being combined with the fourth equation, give the values of these two quantities; and having the value of y, we obtain those of u and x, by means of the first and second equations. The following is the process;

$$x = \frac{39 - 3y}{2}$$

$$5 \times \frac{39 - 3y}{2} - 7z = 11,$$

or
$$195 - 15y - 14z = 22$$

or
$$15y + 14z = 173 \quad (57).$$

The two equations

$$15y + 14z = 173$$
$$4y + 3z = 41,$$

being resolved, give

$$y = 5, \quad z = 7;$$

and by means of these values, we have

$$x = \frac{39 - 3 \times 5}{2} = \frac{39 - 15}{2} = \frac{24}{2}, \quad \text{or} \quad x = 12,$$

$$u = \frac{2 + 2y}{3} = \frac{2 + 10}{3} = \frac{12}{3}, \quad \text{or} \quad u = 4.$$

The numbers sought then are

4, 12, 5 and 7.

82. The method now explained is applicable to literal equations as well as to numerical ones; but the multitude of letters, which it is necessary to employ to represent generally the things given, when the number of equations and unknown quantities exceeds two, has led algebraists to seek for a more simple manner of expressing them. I shall treat of this in the following article; but in order to furnish the reader with the means of exercising himself in putting a problem into an equation, and resolving it, I have subjoined a number of questions, and have placed at the end of each the answer that is required.

1. *A father, being asked the age of his son, said, if from double the age that he is of now, you subtract triple of what he was six years ago, you have his present age.*

Answer. *The child was* 9 *years old.*

2. *Diophantus, the author of the most ancient book on Algebra, that has come down to us, passed a sixth part of his life in infancy, a twelfth part of it in youth; afterward he was married and pass-*

ed in this state a seventh part, and five years more, when he had son, whom he survived four years, and who attained only to ha the age of his father, what was the age of Diophantus when he died

Answer, 84 *years.*

3. *A merchant drew, every year, upon the stock he had in trad the sum of* 1000 *francs for the expense of his family; still his pro erty increased every year, by a third part of what remained after th deduction, and at the end of three years it was doubled; how muc had he at the beginning of the first year?*

Answer, 14800 *francs.*

4. *A merchant has two kinds of tea, the first at* 14 *francs a pound the second at* 18 *francs; how much ought he to take of each to mak up a chest of* 100 *pounds, which should be worth* 1680 *francs?*

Answer, 30 *pounds of the first and* 70 *of the second.*

5. *A person filled, in* 12 *minutes, a vessel containing* 39 *gallons with water, by means of two fountains, which were made to run i succession, and one discharged* 4 *gallons per minute and the othe* 3, *how long did each fountain run?*

Answer, *the first* 3 *minutes, and the second* 9.

6. *At noon the hour and minute hands of a watch are togethe at what point of the dial will they next be in conjunction?*

Answer, *at* 1 *hour* 5 *minutes and* $\frac{5}{11}$.

Obs. This problem refers itself to that of art. 65.

7. *A man, meeting some beggars, wishes to give them* 25 *cents each, but finds upon counting his money, that he wants* 10 *cents i order to do it; he then gives them only* 20 *cents each, and has* 2 *cents left; how much money had he, and what was the number o beggars?*

Answer, 7.

8. *Three brothers purchased an estate for* 50000 *francs, and the first wanted, in order to complete his part of the payment, half of the property of the second; the second would have paid his share with the help of a third of what the first owned, and the third required to make the same payment, in addition to what he had, a fourth part of what the first possessed; what was the amount of each one's property?*

Answer, *the first had* 30000 *francs, the second* 40000, *and the third* 42500.

9. *Three players after a game count their money, one had lost, the other two had gained each as much as he had brought to the play; after the second game, one of the players, who had gained before, lost*

and the two others gained each a sum equal to what he had at the beginning of this second game; at the third game, the player, who had gained till now, lost with each of the others a sum equal to that, which each possessed at the beginning of this last game; they then separated, each having 120 *francs; how much had they each when they commenced playing?*

Answer, *he, who lost at the first game, had* 195 *francs,*
he, who lost at the second 105
he, who lost at the third 60

General formulas for the resolution of equations of the first degree.

83. To obviate the inconvenience referred to in the beginning of the last article, we shall represent all the coefficients of the same unknown quantity by the same letter, but distinguish them by one or more accents, according to the number of equations.

General equations with two unknown quantities are written thus;

$$\begin{aligned} a\,x + b\,y &= c \\ a'\,x + b'\,y &= c'. \end{aligned}$$

The coefficients of the unknown quantity x are both represented by a, those of y by b; but from the accent, which is placed over the letters in the second equation, it may be seen, that they are not considered as having the same value, as the corresponding ones in the first. Thus a' is a quantity different from a, b' a quantity different from b.

If there are three equations, they are expressed;

$$\begin{aligned} a\,x + b\,y + c\,z &= d \\ a'\,x + b'\,y + c'\,z &= d' \\ a''\,x + b''\,y + c''\,z &= d''. \end{aligned}$$

All the coefficients of the unknown quantity x are designated by the letter a, those of y by b, those of z by c; but the several letters are distinguished by different accents, which show, that they denote different quantities. Thus a, a', a'', are three different quantities. The same may be said of b, b', b'', &c.

Following this method, if we have four unknown quantities, and four equations, we may write them thus;

$$\begin{aligned} a\,x + b\,y + c\,z + d\,u &= e \\ a'\,x + b'\,y + c'\,z + d'\,u &= e' \\ a''\,x + b''\,y + c''\,z + d''\,u &= e'' \\ a'''\,x + b'''\,y + c'''\,z + d'''\,u &= e'''. \end{aligned}$$

84. To avoid fractions, and simplify the calculation, we may vary the process of elimination in the following manner.

Let there be the equations

$$a\,x + b\,y = c$$
$$a'\,x + b'\,y = c';$$

it is evident, that if one of the unknown quantities, x, for example, has the same coefficient in the two equations, we have only to subtract one of these equations from the other, in order to make this unknown quantity to disappear. This may be seen at once in the equations

$$10\,x + 11\,y = 27,$$
$$10\,x + 9\,y = 15,$$

which give

$$11\,y - 9\,y = 27 - 15, \text{ or } 2\,y = 12, \text{ or } y = 6.$$

It is evident, that the coefficients of x may be immediately made equal in the equations

$$a\,x + b\,y = c$$
$$a'\,x + b'\,y = c'$$

by multiplying the two members of the first by a', the coefficient of x in the second, and the two members of the second by a, the coefficient of x in the first; we thus obtain,

$$a\,a'\,x + a'\,b\,y = a'\,c$$
$$a\,a'x + a\,b'\,y = a\,c'.$$

Then subtracting the first of these from the second, the unknown quantity x disappears; and we have

$$(a\,b' - a'\,b)\,y = a\,c' - a'\,c,$$

an equation, which contains only the unknown quantity y; from this we may deduce,

$$y = \frac{a\,c' - c\,a'}{a\,b' - b\,a'}.$$

The method, we have just employed, may always be applied to equations of the first degree, to exterminate any one of the unknown quantities.

By exterminating, in the same manner, the unknown quantity y, we may find the value of x.

If we apply this process to three equations, containing x, y and z, we may first exterminate x from the first and second, then from the first and third; we thus obtain two equations, which contain only y and z, from which we may exterminate y.

When this calculation is performed, the equation containing z,

to which we arrive, will have a factor common to all its terms, and consequently will not be the most simple, which may be obtained.

85. Bézout has given a very simple method for exterminating at once all the unknown quantities except one, and for reducing the question immediately to equations, which contain one unknown quantity less, than the equations proposed. Although this process is necessary, only when equations with three unknown quantities are employed, we shall, in order to give a complete view of the subject, begin by applying it to those, which contain only two.

Let there be the equations

$$a x + b y = c$$
$$a' x + b' y + c';$$

multiplying the first by any indeterminate quantity m, we have

$$a m x + b m y = m c;$$

subtracting from this result the equation

$$a' x + b' y = c',$$

there remains

$$a m x - a' x + b m y - b' y = c m - c',$$
$$\text{or } (a m - a') x + (b m - b') y = c m - c'.$$

Since m is an indeterminate quantity, we may suppose it to be such, that $b m = b'$. In this case, the term multiplied by y disappears, and we have

$$x = \frac{c m - c'}{a m - a'};$$

but since $b m = b'$, it follows that,

$$m = \frac{b'}{b};$$

therefore

$$x = \frac{\frac{c b'}{b} - c'}{\frac{a b'}{b} - a'} = \frac{c b' - b c'}{a b' - b a'}.$$

If, instead of supposing $b m = b'$, we make $a m = a'$, the term, which contains x, will vanish, and we shall have

$$y = \frac{c m - c'}{b m - b'}.$$

The value of m will not be the same as before; for we shall have

$$m = \frac{a'}{a};$$

and by substituting this in the expression for y, we find

$$y=\frac{ca'-ac'}{ba'-ab'}.$$

If we change the signs of the numerator and denominator of this value of y, the denominator will become the same, as that in the expression for x, since we shall have

$$y=\frac{ac'-ca'}{ab'-ba'}.$$

86. Next let there be the three equations

$$\begin{aligned} a\,x+b\,y+c\,z&=d\\ a'\,x+b'\,y+c'\,z&=d'\\ a''x+b''y+c''z&=d''; \end{aligned}$$

we shall be led, by an obvious analogy, to multiply the first of these equations by m, and the second by n, m and n being indeterminate quantities, to add together the results, and from the sum to subtract the third; by this means, all the equations will be employed at the same time, and the two new quantities m and n, which we may dispose of, as we please, will admit of any determinate value, which may be necessary to make both the unknown quantities to disappear in the result. Having proceeded in this manner, and united the terms by which the same unknown quantity is multiplied, we shall have

$$(am+a'n-a'')\,x+(bm+b'n-b'')\,y+(cm+c'n-c'')\,z$$
$$=dm+d'n-d''.$$

If we would make the unknown quantities x and y to disappear, we must take the equations

$$\begin{aligned} am+a'n&=a''\\ bm+b'n&=b'', \end{aligned}$$

and then we obtain

$$z=\frac{dm+d'n-d''}{cm+c'n-c''}.$$

From the two equations, in which m and n are the unknown quantities, it is easy to deduce the value of these quantities, by means of the results obtained in the preceding article; for it is only necessary to change in these results x into m, y into n, and to write instead of the letters

$$\left.\begin{matrix} a, & b, & c \\ a', & b', & c' \end{matrix}\right\}\text{ the letters }\left\{\begin{matrix} a, & a', & a'' \\ b, & b', & b'', \end{matrix}\right.$$

which gives

$$m=\frac{a''b'-b''a'}{a\,b'-b\,a'}$$

$$n=\frac{a\,b''-b\,a''}{a\,b'-b\,a'}.$$

Substituting these values in the expression for z, and reducing all the terms to the same denominator, we have, (a)

$$z=\frac{d\,(b'a''-a'b'')+d'\,(a\,b''-b\,a'')-d''\,(a\,b'-b\,a')}{c\,(b'a''-a'b'')+c'\,(a\,b''-b\,a'')-c''\,(a\,b'-b\,a')}.$$

If we had made the terms containing x and z to disappear, we should have had y; the letters m and n would have depended upon the equations

$$a\,m+a'\,n=a'' \qquad c\,m+c'\,n=c'',$$

and proceeding as before, we should have found

$$y=\frac{d\,(c'a''-a'c'')+d'\,(a\,c''-c\,a'')-d''\,(a\,c'-c\,a')}{b\,(c'a''-a'c'')+b'\,(a\,c''-c\,a'')-b''\,(a\,c'-c\,a')}.$$

Lastly, by assuming the equations

$$b\,m+b'\,n=b'', \qquad c\,m+c'\,n=c'',$$

we make the terms multiplied by y and z to disappear; and we have

$$x=\frac{d\,(c'b''-b'c'')+d'\,(b\,c''-c\,b'')-d''\,(b\,c'-c\,b')}{a\,(c'b''-b'c'')+a'\,(b\,c''-c\,b'')-a''\,(b\,c'-c\,b')}.$$

These values being developed in such a manner, as to make the terms alternately positive and negative, if we change, at the same time, the signs of the numerator and denominator, in the first and third, we shall give them the following forms;

$$z=\frac{a\,b'\,d''-a\,d'\,b''+d\,a'\,b''-b\,a'\,d''+b\,d'\,a''-d\,b'\,a''}{a\,b'\,c''-a\,c'\,b''+c\,a'\,b''-b\,a'\,c''+b\,c'\,a''-c\,b'\,a''},$$

$$y=\frac{a\,d'\,c''-a\,c'\,d''+c\,a'\,d''-d\,a'\,c''+d\,c'\,a''-c\,d'\,a''}{a\,b'\,c''-a\,c'\,b''+c\,a'\,b''-b\,a'\,c''+b\,c'\,a''-c\,b'\,a''},$$

$$x=\frac{d\,b'\,c''-d\,c'\,b''+c\,d'\,b''-b\,d'\,c''+b\,c'\,d''-c\,b'\,d''}{a\,b'\,c''-a\,c'\,b''+c\,a'\,b''-b\,a'\,c''+b\,c'\,a''-c\,b'\,a''}.$$

87. Let there be the four equations

$$\begin{aligned}
a\,x+b\,y+c\,z+d\,u&=e\\
a'\,x+b'\,y+c'\,z+d'\,u&=e'\\
a''x+b''y+c''z+d''u&=e''\\
a'''x+b'''y+c'''z+d'''u&=e''';
\end{aligned}$$

(a)
$$z=\frac{d\,\dfrac{a''b'-b''a'}{a\,b'-b\,a'}+d'\,\dfrac{a\,b''-b\,a''}{a\,b'-b\,a'}-d''\,\dfrac{a\,b'-b\,a'}{a\,b'-b\,a'}}{c\,\dfrac{a''b'-b''a'}{a\,b'-b\,a'}+c'\,\dfrac{a\,b''-b\,a''}{a\,b'-b\,a'}-c''\,\dfrac{a\,b'-b\,a'}{a\,b'-b\,a'}}$$

if we multiply the first by m, the second by n, the third by p, and from the sum of their products subtract the fourth, we shall have

$$(am+a'n+a''p-a''')x+(bm+b'n+b''p-b''')y$$
$$+(cm+c'n+c''p-c''')z+(dm+d'n+d''p-d''')u$$
$$=em+e'n+e''p-e'''.$$

In order to obtain u, we make

$$am+a'n+a''p=a'''$$
$$bm+b'n+b''p=b'''$$
$$cm+c'n+c''p=c''',$$

we then have

$$u=\frac{em+e'n+e''p-e'''}{dm+d'n+d''p-d'''}.$$

The preceding equations, which must give m, n, and p, may be resolved by means of the formulas found for the case of three unknown quantities. This method will appear very simple and convenient; but the nature of the results obtained above will furnish us with a rule for finding them without any calculation.

88. To begin with the most simple case, we take an equation with one unknown quantity, $ax=b$; from this we find

$$x=\frac{b}{a},$$

in which the numerator is the whole known term b, and the denominator the coefficient a of the unknown quantity.

From the two equations

$$ax+by=c, \qquad a'x+b'y=c',$$

we have already deduced

$$x=\frac{cb'-bc'}{ab'-ba'} \qquad y=\frac{ac'-ca'}{ab'-ba'}.$$

The denominator in this case is composed also of the letters a, a', b, b', by which the unknown quantities are multiplied. We first write a by the side of b, which gives ab; we then change the order of a and b, and obtain ba; prefixing to this the sign $-$ we have $ab-ba$; lastly we place an accent over the last letter in each term, and the expression becomes $ab'-ba'$ for the denominator.

From this expression we may find the numerator. To obtain that for x, we have only to change each a into c, and each b into c for that of y, putting an accent over the last letter as before; in this way we find $cb'-bc'$ for the one, and $ac'-ca'$ for the

her. *The numerator may therefore be found from the denominator,* well in cases where there are two unknown quantities, as when ere is only one, *by changing the coefficient of the unknown quany sought, into the known term or second member, and retaining the cents, which belonged to the coefficients.*

The same rule may be applied to equations with three unown quantities, as we shall see by merely inspecting the vals, which result from these equations. With respect to the deminator, it is necessary further to illustrate the method by hich it is formed. Now, since in the case of two unknown antities, the denominator presents all the possible tranpositions the letters a and b, by which the unknown quantities are mullied, it may be supposed, that when there are three unknown antities, their denominator will contain all the arrangements the three letters a, b, c. These arrangements may be formed the following manner.

We first make the transpositions $a\,b - b\,a$ with the two letters and b, then after the first term $a\,b$, write the third letter c, which ves $a\,b\,c$; making this letter pass through all the places, obrving each time to change the sign, and not to derange the orr in which a and b respectively stand, we obtain

$$a\,b\,c - a\,c\,b + c\,a\,b.$$

roceeding in the same manner with respect to the second term $b\,a$, we find

$$-b\,a\,c + b\,c\,a - c\,b\,a;$$

nnecting these products with the preceding, and placing over e second letter one accent, and over the third two, we have

$$a\,b'\,c'' - a\,c'\,b'' + c\,a'\,b'' - b\,a'\,c'' + b\,c'\,a'' - c\,b'\,a'',$$

result, which agrees with that presented by the formulas, obined above.

From this it is obvious, that, in order to form a denominator the case of four unknown quantities, it is necessary to introce the letter d into each of the six products

$$a\,b\,c - a\,c\,b + c\,a\,b - b\,a\,c + b\,c\,a - c\,b\,a,$$

d to make it occupy successively all the places. The term c, for example, will give the four following;

$$a\,b\,c\,d - a\,b\,d\,c + a\,d\,b\,c - d\,a\,b\,c.$$

we observe the same method in regard to the five other procts, the whole result will be twenty four terms, in each of

which, the second letter will have one accent, the third two, and the fourth three. The numerators of the unknown quantities u, z, y and x, are found by the rule already given.*

89. We may employ these formulas for the resolution of numerical equations. In doing this, we must compare the terms of the equations proposed with the corresponding terms of the general equations, given in the preceding articles.

To resolve, for example, the three equations

$$\begin{aligned} 7x+5y+2z&=\ 79\\ 8x+7y+9z&=122\\ x+4y+5z&=\ 55, \end{aligned}$$

it is necessary to compare the terms with those of the equations given in art. 86. We have then

$$\begin{aligned} a&=7,\ b=5,\ c=2,\ d=\ 79\\ a'&=8,\ b'=7,\ c'=9,\ d'=122\\ a''&=1,\ b''=4,\ c''=5,\ d''=\ 55. \end{aligned}$$

Substituting these values in the general expressions for the unknown quantities x, y and z, and going through the operations, which are indicated, we find

$$x=4,\qquad y=9,\qquad z=3.$$

It is important to remark, that the same expressions may be employed, even when the proposed equations are not, in all their terms, affected with the sign $+$, as the general equations from which these expressions are deduced appear to require. If we have, for example,

$$\begin{aligned} 3x-9y+8z&=\ 41\\ -5x+4y+2z&=-20\\ 11x-7y-6z&=\ 37, \end{aligned}$$

in comparing the terms of these equations with the corresponding ones in the general equations, we must attend to the signs, and the result will be

$$\begin{aligned} a&=+\ 3,\ b=-9,\ c=+8,\ d=+41\\ a'&=-\ 5,\ b'=+4,\ c'=+2,\ d'=-20\\ a''&=+11,\ b''=-7,\ c''=-6,\ d''=+37. \end{aligned}$$

We are then to determine by the rules given in art. 31, the sign,

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* M. Laplace, in the second part of the Mémoires de l'Académie des Sciences for 1772, p. 294, has demonstrated these rules *à priori*. See also *Annales de Mathématiques pures appliquées*, by M. Gergonne, vol. iv. p. 148.

which each term of the general expressions for x, y and z ought to have, according to the signs of the factors of which it is composed. Thus we find, for example, that the first term of the common denominator, which is $a\,b'\,c''$, becoming $+3 \times +4 \times -6$, changes the sign of the product, and gives -72. If we observe the same method with respect to the other terms, both of the numerators and denominators, taking the sum of those, which are positive, and also of those which are negative, we obtain

$$x = \frac{2874 - 2834}{592 - 622} = \frac{-60}{-30} = +2$$

$$y = \frac{3022 - 2932}{592 - 622} = \frac{+90}{-30} = -3.$$

$$z = \frac{3859 - 3889}{592 - 622} = \frac{-30}{-30} = +1.$$

Equations of the second degree, having only one unknown quantity.

90. Hitherto I have been employed upon equations of the *first degree*, or such as involve only the first power of the unknown quantities; but were the question proposed, *To find a number, which, multiplied by five times itself, will give a product equal to* 125; if we designate this number by x, five times the same will be $5\,x$, and we shall have

$$5\,x^2 = 125.$$

This is an equation of the *second degree*, because it contains x^2, or the second power of the unknown quantity. If we free this second power from its coefficient 5, we obtain

$$x^2 = \frac{125}{5}, \qquad \text{or} \qquad x^2 = 25.$$

We cannot here obtain the value of the unknown quantity x as in art. 11, and the question amounts simply to this, to find a number which, multiplied by itself, will give 25. It is obvious that this number is 5; but it seldom happens that the solution is so easy; hence arises this new numerical question; *to find a number, which, multiplied by itself, will give a product equal to a proposed number*; or, which is the same thing, from the second power of a number, to retrace our steps to the number from which it is derived, and which is called the *square root*. I shall proceed in the first place to resolve this question, as it is involved in the determination of the unknown quantities, in all equations of the second degree.

91. The method employed in finding or *extracting* the roots of numbers, supposes the second power of such, as are expressed by only one figure to be known. See the nine primitive numbers with their second powers written under them respectively.

1	2	3	4	5	6	7	8	9
1	4	9	16	25	36	49	64	81

It is evident from this table, that the second power of a number expressed by one figure, contains only two figures; 10, which is the least number expressed by two figures, has for its square a number composed of three, 100. In order to resolve the second power of a number consisting of two figures, we must attend to the method by which it is formed; for this purpose we must inquire, how each part of the number 47, for example, is employed in the production of the square of this number.

We may resolve 47 into $40+7$, or into 4 tens and 7 units; if we represent the tens of the proposed number by a, and the units by b, the second power will be expressed by

$$(a+b)(a+b)=a^2+2ab+b^2;$$

that is, it is made up of three parts, namely, *the square of the tens, twice the product of the tens multiplied by the units, and the square of the units.* In the example we have taken, $a=4$ tens or 40 units, and $b=7$; we have then

$$\begin{aligned} a^2 &= 1600 \\ 2ab &= 560 \\ b^2 &= 49 \end{aligned}$$

$$\text{Total,}\quad a^2+2ab+b^2=2209=47\times 47.$$

Now in order to return, by a reverse process, from the number 2209 to its root, we may observe, that the square of the tens, 1600, has no figure, which denotes a rank inferior to hundreds, and that it is the greatest square, which the 22 hundreds, comprehended in 2209, contain; for 22 lies between 16 and 25, that is, between the square of 4 and that of 5, as 47 falls between 4 tens or 40, and 5 tens or 50.

We find therefore, upon examination, that the greatest square contained in 22 is 16, the root of which 4 expresses the number of tens in the root of 2209; subtracting 16 hundreds or 1600 from 2209, the remainder 609 contains double the product of the tens by the units, 560, and the square of the units 49. But as double the product of the tens by the units has no figure infe-

rior to tens, it must be found in the two first figures 60 of the remainder 609, which contain also the tens, arising from the square of the units. Now if we divide 60 by double of the tens 8, and neglect the remainder, we have a quotient 7 equal to the units sought. If we multiply 8 by 7, we have double the product of the tens by the units, 560; subtracting this from the whole remainder 609, we obtain a difference 49, which must be, and in fact is the square of the units.

This process may be exhibited thus;

$$\begin{array}{r|l}
22{,}09 & 47 \\ \hline
16 & 87 \\ \hline
60{,}9 & \\
60\ 9 & \\ \hline
000 &
\end{array}$$

We write the proposed number in the manner of a dividend, and assign for the root the usual place of the divisor. We then separate the units and tens by a comma, and employ only the two first figures on the left, which contain the square of the tens found in the root. We seek the greatest square 16, contained in these two figures, put the root 4 in its assigned place, and subtract 16 from 22. To the remainder we bring down the two other figures, 09, of the proposed number, separating the last, which does not enter into double the product of the tens by the units, and divide the remainder on the left by 8, double the tens in the root, which gives for the quotient the units 7. In order to collect into one expression the two last parts of the square contained in 609, we write 7 by the side of 8, which gives 87, equal to double the tens plus the units, or $2a+b$; this multiplied by 7 or b, reproduces $609 = 2ab + b^2$, or double the product of the tens by the units, plus the square of the units. This being subtracted leaves no remainder, and the operation shows, that 47 is the square root of 2209.

If it were required to extract the square root of 324; the operation would be as follows;

$$\begin{array}{r|l}
3{,}24 & 18 \\
1 & \\ \hline
22{,}4 & 28 \\
22\ 4 & \\ \hline
000 &
\end{array}$$

Proceeding as in the last example, we obtain 1 for the place of tens of the root; this doubled gives the number 2, by which the two first figures 22 of the remainder are to be divided. Now 22 contains 2 eleven times, but the root can neither be more than 10, nor 10; even 9 is in fact too large, for if we write 9 by the side of 2, and multiply 29 by 9, as the rule requires, the result is 261, which cannot be subtracted from 224. We are therefore to consider the division of 22 by 2, only as a means of approximating the units, and it becomes necessary to diminish the quotient obtained, until we arrive at a product, which does not exceed the remainder 224. The number 8 answers to this condition, since $8 \times 28 = 224$; therefore the root sought is 18.

By resolving the square of 18 into its three parts we find;

$$\begin{array}{lr} & a^2 = 100 \\ & 2ab = 160 \\ & b^2 = 64 \\ \hline \text{Total,} & 324 = 18 \times 18, \end{array}$$

and it may be seen, that the 6 tens, contained in the square of the units, being united to 160, double the product of the tens by the units, alters this product in such a manner, that a division of it by double the tens will not give exactly the units.

92. It will not be difficult, after what has been said, to extract the square root of a number, consisting of three or four figures; but some further observations, founded upon the principles above laid down, may be necessary to enable the reader to extract the root of any number whatever.

No number less than 100 can have a square consisting of more than four figures, since that of 100 is 10000, or the least number expressed by five figures. In order therefore to analyze the square of any number exceeding 100, of 473, for example, we may resolve it into $470 + 3$, or 47 tens plus 3 units. To obtain its square from the formula

$$a^2 + 2ab + b^2,$$

we make $a = 47$ tens $= 470$ units, $b = 3$ units, then

$$\begin{array}{lr} & a^2 = 220900 \\ & 2ab = 2820 \\ & b^2 = 9 \\ \hline \text{Total,} & 223729 = 473 \times 473. \end{array}$$

In this example, it is evident that the square of the tens has no

figure inferior to hundreds, and this is a general principle, since tens multiplied by tens, always give hundreds (*Arith.* 32).

It is therefore in the part 2237, which remains on the left of the proposed number, after we have separated the tens and units, that it is necessary to seek the square of the tens; and as 473 lies between 47 tens, or 470, and 48 tens, or 480, 2237 must fall between the square of 47 and that of 48; hence the greatest square contained in 2237, will be the square of 47, or that of the tens of the root. In order to find these tens, we must evidently proceed, as if we had to extract the square root of 2237 only; but instead of arriving at an exact result, we have a remainder, which contains the hundreds arising from double the product of the 47 tens multiplied by the units.

The operation is as follows

22,37,29	473
16	87
63,7	943
60 9	
282,9	
282 9	
0	

We first separate the two last figures 29, and in order to extract the root of the number 2237, which remains on the left, we further separate the two last figures 37 of this number; the proposed number is then divided into portions of two figures, beginning on the right and advancing to the left. Proceeding with the two first portions as in the preceding article, we find the two first figures 47 of the root; but we have a remainder 28, which, joined to the two figures 29 of the last portion, contains double the product of the 47 tens by the units, and the square of the units. We separate the figure 9, which forms no part of double the product of the tens by the units, and divide 282 by 94, double the 47 tens; writing the quotient 3 by the side of 94, and multiplying 943 by 3, we obtain 2829, a number exactly equal to the last remainder, and the operation is completed.

93. In order to show, by what method we are to proceed with any number of figures, however great, I shall extract the root of

22391824. Whatever this root may be, we may suppose it capable of being resolved into tens and units, as in the preceding examples. As the square of the tens has no figure inferior to hundreds, the two last figures 24 cannot make a part of it; we may therefore separate them, and the question will be reduced to this, to find the greatest square contained in the part 223918, which remains on the left. This part consisting of more than two figures, we may conclude, that the number, which expresses the tens in the root sought, will have more than one figure; it may therefore be resolved, like the others, into tens and units. As the square of the tens does not enter into the two last figures 18 of the number 223918, it must be sought in the figures 2239, which remain on the left; and since 2239 still consists of more than two figures, the square, which is contained in it must have a root, which consists of at least two; the number which expresses the tens sought will therefore have more than one figure; it is then, lastly, in 22 that we must seek the square of that, which represents the units of the highest place in the root required. By this process, which may be extended to any length we please, the proposed number may be divided into portions of two figures from right to left; it must be understood however, that the last figure on the left may consist of only one figure.

Having divided the proposed number into portions as below, we proceed with the three first portions, as in the preceding article; and when we have found the three first figures 473 of the root, to the remainder 189 we bring down the fourth portion 24, and consider the number 18924, as containing double the product of the 473 tens already found by the units sought, plus the square of these units. We separate the last figure 4; divide those, which remain on the left, by 946, double of 473, and then make trial of the quotient 2, as in the preceding examples.

22,39,18,24	4732
16	87
63,9	943
60 9	9462
301,8	
282 9	
1892,4	
1892 4	
0000 0	

Here the operation, in the present case, terminates; but it is very obvious, that if we had one portion more, the four figures already found 4732, would express the tens of a root, the units of which would remain to be sought; we should proceed therefore

to divide the remainder now found, together with the first figure of the following portion, by double of these tens, and so on for each of the portions to be successively brought down.

94. If, after having brought down a portion, the remainder, joined to the first figure of this portion, does not contain double of the figures already found, a cypher must be placed in the root; for the root, in this case, will have no units of this rank; the following portion is then to be brought down, and the operation to be continued as before. The example subjoined will illustrate this case. The quantities to be subtracted are not put down, but the subtractions are supposed to be performed mentally, as in division.

49,42,09	703
04,20, 9	1403
0 00 0	

95. Every number, it will be perceived, is not a perfect square. If we look at the table given, page 100, we shall see that between the squares of each of the nine primitive numbers, there are intervals comprehending many numbers, which have no assignable root; 45, for instance, is not a square, since it falls between 36 and 49. It very often happens, therefore, that the number, the root of which is sought, does not admit of one; but if we attempt to find it, we obtain for the result the root of the greatest square, which the number contains. If we seek, for example, the root of 2276, we obtain 47, and have a remainder 67, which shows, that the greatest square, contained in 2276, is that of 47 or 2209.

As a doubt may sometimes arise, after having obtained the root of a number, which is not a perfect square, whether the root found be that of the greatest square contained in the number, I shall give a rule, by which this may be readily determined. As the square of $a + b$ is

$$a^2 + 2ab + b^2,$$

if we make $b = 1$, the square of $a + 1$ will be

$$a^2 + 2a + 1,$$

a quantity which differs from a^2, the square of a, by double of a, plus unity. Therefore *if the root found can be augmented by unity, or more than unity, its square, subtracted from the proposed number, will leave a remainder at least equal to twice this root plus unity.* Whenever this is not the case, the root obtained will be, in fact, that of the greatest square contained in the number proposed.

96. Since a fraction is multiplied by another fraction, when their numerators are multiplied together, and their denominators

together, it is evident that the product of a fraction multiplied by itself, or *the square of a fraction is equal to the square of its numerator, divided by the square of its denominator*. Hence it follows that to extract *the square root of a fraction*, we extract the square *root of its numerator and that of its denominator.* Thus the root of [illegible] is [illegible], because 5 is the square root of [illegible] and [illegible] that of [illegible]

It is very important to remark, that not only are the squares of fractions, properly so called, always fractions, but every *fractional number, which is irreducible*, (Arith. [illegible]) *will, when multiplied by itself, give a fractional result, which is also irreducible.*

97. This proposition depends upon the following; *Every prime number* P, *which will divide the product* A B *of two numbers* A *and* B, *will necessarily divide one of these numbers.*

Let us suppose, that it will not divide B, and that B is greater; if we designate the entire part of the quotient by q, and the remainder by B', we have

$$B = q\,P + B',$$

multiplying by A, we obtain

$$A\,B = q\,A\,P + A\,B',$$

and dividing the two members of this equation by P, we have

$$\frac{A\,B}{P} = q\,A + \frac{A\,B'}{P};$$

from which it appears, that if $A\,B$ be divisible by P, the product $A\,B'$ will be divisible by the same number. Now B', being the remainder after the division of B by P, must be less than P; therefore B' cannot be divided by P; if we divide P by B', we have a quotient q' and a remainder B''; if further we divide B' by B'', we have a quotient q'' and a remainder B''', and so on, since P is a prime number.

We have therefore the following series of equations;

$$P = q'\,B' + B'', \quad P = q''\,B'' + B''', \text{ \&c.}$$

multiplying each of these by A, we obtain

$$A\,P = q'\,A\,B' + A\,B'', \quad A\,P = q''\,A\,B'' + A\,B''' \text{ \&c.}$$

dividing by P, we have

$$A = q'\,\frac{A\,B'}{P} + \frac{A\,B''}{P}, \quad A = q''\,\frac{A\,B''}{P} + \frac{A\,B'''}{P}, \text{ \&c.}$$

From these results it is evident, that if $A\,B'$ be divisible by P, the products $A\,B''$, $A\,B'''$, &c. will also be divisible by it. But the remainders B', B'', B''', &c. are becoming less and less continually

till they finally terminate in unity, for the operation exhibited above may be continued in the same manner, while the remainder is greater than 1, since P is a prime number. Now when the remainder becomes unity, we have the product $A \times 1$, which must be divisible by P; therefore A also must be divisible by P.

Hence, if the prime number P, which we have supposed not to divide B, will not divide A, it will not divide the product of these numbers.

(*This demonstration is taken principally from the* Théorie des nombres of M. Legendre.)

98. Now when the fraction $\frac{b}{a}$ is irreducible, there is no prime number, which will divide, at the same time, b and a; but from the preceding demonstration, it is evident, that every prime number, which will not divide a, will not divide $a \times a$, or a^2, every prime number, which will not divide b, will not divide $b \times b$, or b^2; the numbers a^2 and b^2 are therefore, in this case, prime to each other; and consequently the square $\frac{b^2}{a^2}$ of the fraction $\frac{b}{a}$, being irreducible, as well as the fraction itself, cannot become an entire number (B).

99. From this last proposition it follows, that *entire numbers, except only such, as are perfect squares, admit of no assignable root, either among whole numbers or fractions.* Yet it is evident, that there must be a quantity, which, multiplied by itself, will produce any number whatever, 2276, for instance, and that, in the present case, this quantity lies between 47 and 48; for 47×47 gives a product less than this number, and 48×48 gives one greater. Dividing then the difference between 47 and 48 by means of fractions, we may obtain numbers that, multiplied by themselves, will give products greater than the square of 47, but less than that of 48, and which will approach nearer and nearer to the number 2276.

The extraction of the square root, therefore applied to numbers, which are not perfect squares, makes us acquainted with a new species of numbers, in the same manner, as division gives rise to fractions; but there is this difference between fractions and the roots of numbers, which are not perfect squares; that the former, which are always composed of a certain number of parts of unity, have with unity a *common measure*, or a rela-

tion which may be expressed by whole numbers, which the latter have not.

If we conceive unity to be divided into five parts, for example, we express the quotient arising from the division of 9 by 5, or $\frac{9}{5}$, by nine of these parts; $\frac{1}{5}$ then, being contained five times in unity, and nine times in $\frac{9}{5}$, is the *common measure* of unity and the fraction $\frac{9}{5}$, and the relation these quantities have to each other is that of the entire numbers 5 and 9.

Since whole numbers, as well as fractions, have a common measure with unity, we say that these quantities are *commensurable* with unity, or simply that they are *commensurable*; and since their *relations* or *ratios*, with respect to unity, are expressed by entire numbers, we designate both whole numbers and fractions, by the common name of *rational numbers*.

On the contrary, the square root of a number, which is not a perfect square, is *incommensurable* or *irrational*, because, as it cannot be represented by any fraction, into whatever number of parts we suppose unity to be divided, no one of these parts will be sufficiently small to measure exactly, at the same time, both this root and unity.

In order to denote, in general, that a root is to be extracted, whether it can be exactly obtained or not, we employ the character $\sqrt{}$, which is called a *radical* sign;

$\sqrt{16}$ is equivalent to 4,

$\sqrt{2}$ is *incommensurable* or *irrational*.

100. Although we cannot obtain, either among whole numbers or fractions, the exact expression for $\sqrt{2}$, yet we may approximate it, to any degree we please, by converting this number into a fraction, the denominator of which is a perfect square. The root of the greatest square contained in the numerator will then be that of the proposed number expressed in parts, the value of which will be denoted by the root of the denominator.

If we convert, for example, the number 2 into twenty-fifths, we have $\frac{50}{25}$. As the root of 50 is 7, so far as it can be expressed in whole numbers, and the root of 25 exactly 5, we obtain $\frac{7}{5}$, or $1\frac{2}{5}$ for the root of 2, to within one fifth.

101. This process, founded upon what was laid down in article 96, that the square of a fraction is expressed by the square of the numerator divided by the square of the denominator, may evidently be applied to any kind of fraction whatever, and more readily

to decimals than to others. It is manifest, indeed, from the nature of multiplication, that the square of a number expressed by tenths will be hundredths, and that the square of a number expressed by hundredths will be ten thousandths, and so on; and consequently, that *the number of decimal figures in the square is always double that of the decimal figures in the root.* The truth of this remark is further evident from the rule observed in the multiplication of decimal numbers, which requires that a product should contain as many decimal figures, as there are in both the factors. In any assumed case therefore, the proposed number, considered as the product of its root multiplied by itself, must have twice as many decimal figures as its root.

From what has been said it is clear, that in order to obtain the square root of 227, for example, to within one hundredth, it is necessary to reduce this number to ten thousandths, that is, to annex to it four cyphers, which gives 2270000 ten thousandths. The root of this may be extracted in the same manner, as that of an equal number of units; but to show that the result is hundredths, we separate the two last figures on the right by a comma. We thus find that the root of 227 is 15,06, accurate to hundredths. The operation may be seen below;

2,27,00.00	1506
12,7	25
2 00 00	3006
19 64	

If there are decimals already in the proposed number, they should be made even. To extract, for example, the root of 51,7 we place one cypher after this number, which makes it hundredths; we then extract the root of 51,70. If we proposed to have one decimal more, we should place two additional cyphers after this number, which would give 51,7000; we should then obtain 7,19 for the root.

If it were required to find the square root of the numbers 2 and 3 to seven places of decimals, we should annex fourteen cyphers to these numbers; the result would be

$$\sqrt{2} = 1{,}4142136, \qquad \sqrt{3} = 1{,}7320508.$$

102. When we have found more than half the number of figures, of which we wish the root to consist, we may obtain the rest simply by division. Let us take, for example, 32976; the square root of this number is 181, and the remainder 215. If we divide this

remainder 215 by 362, double of 181, and extend the quotient to two decimal places, we obtain 0,59, which must be added to 181; the result will be 181,59 for the root of 32976, which is accurate to within one hundredth.

In order to prove that this method is correct, let us designate the proposed number by N, the root of the greatest square contained in this number by a, and that which it is necessary to add to this root to make it the exact root of the proposed number by b; we have then

$$N = a^2 + 2ab + b^2,$$

from which we obtain

$$N - a^2 = 2ab + b^2;$$

dividing this by $2a$, we find

$$\frac{N - a^2}{2a} = b + \frac{b^2}{2a}.$$

From this result it is evident, that the first member may be taken for the value of b, so long as the quantity $\frac{b^2}{2a}$ is less than a unit of the lowest place found in b. But as the square of a number cannot contain more than twice as many figures as the number itself, it follows, that if the number of figures in a exceeds double those in b, the quantity $\frac{b^2}{2a}$ will then be a fraction.

In the preceding example, $a = 181$ units, or 18100 hundredths, and consequently contains one figure more than the square of 59 hundredths; the fraction then $\frac{b^2}{2a}$ becomes, in this case, $\frac{(59)^2}{2 \times 18100} = \frac{3481}{36200}$, and is less than a unit of the second part 59, or than a hundredth of a unit of the first.

103. This leads to a method of approximating the square root of a number by means of vulgar fractions. It is founded on the circumstance that a, being the root of the greatest square contained in N, b is necessarily a fraction, and $\frac{b^2}{2a}$ being much smaller than b may be neglected.

If it were required, for example, to extract the square root of 2; as the greatest square contained in this number is 1, if we subtract this, we have a remainder 1. Dividing this remainder by double of the root we obtain $\frac{1}{2}$; taking this quotient for the value of the quantity b, we have, for the first approximation to the root, $1 + \frac{1}{2}$, or $\frac{3}{2}$. Raising this root to its square, we

find $\frac{9}{4}$, which, subtracted from 2 or $\frac{8}{4}$, gives for a remainder $-\frac{1}{4}$. In this case the formula

$$\frac{N-a^2}{2a}=b+\frac{b^2}{2a}$$

becomes

$$-\frac{1}{12}=b+\frac{b^2}{2a}.$$

Substituting $-\frac{1}{12}$ for b, we have for the second approximation $\frac{3}{2}-\frac{1}{12}=\frac{17}{12}$; taking the square of $\frac{17}{12}$, we find $\frac{289}{144}$, a quantity, which still exceeds 2 or $\frac{288}{144}$. Substituting $\frac{17}{12}$ for a, we obtain

$$-\frac{1}{12\times 34}=b+\frac{b^2}{2a};$$

which gives

$$b=-\frac{1}{12\times 34}=-\frac{1}{408};$$

the third approximation will then be

$$\frac{17}{12}-\frac{1}{12\times 34}=\frac{17\times 34-1}{408}=\frac{577}{408}.$$

This operation may be easily continued to any extent we please. I shall give, in the *Supplement* to this treatise, other formulas more convenient for extracting roots in general.

104. In order to approximate the square root of a fraction, the method, which first presents itself is, to extract, by approximation the square root of the numerator and that of the denominator; but with a little attention it will be seen, that we may avoid one of these operations by making the denominator a perfect square. This is done by multiplying the two terms of the proposed fraction by the denominator. If it were required, for example, to extract the square root of $\frac{3}{7}$, we might change this fraction into

$$\frac{3\times 7}{7\times 7}=\frac{21}{49},$$

by multiplying its two terms by the denominator 7. Taking the root of the greatest square contained in the numerator of this fraction, we have $\frac{4}{7}$ for the root of $\frac{3}{7}$, accurate to within $\frac{1}{7}$.

If a greater degree of exactness were required, the fraction $\frac{3}{7}$ must be changed by approximation or otherwise into another, the denominator of which is the square of a greater number than 7. We shall have, for example, the root sought within $\frac{1}{15}$, if we convert $\frac{3}{7}$ into 225ths, since 225 is the square of 15; thus the fraction becomes $\frac{675}{7}$ of one 225th, or $\frac{96}{225}$, within $\frac{1}{225}$; the root

of $\frac{96}{225}$ falls between $\frac{9}{15}$ and $\frac{10}{15}$, but approaches nearer to the second fraction than to the first, because 96 approaches nearer to a hundred than to 81; we have then $\frac{10}{15}$ or $\frac{2}{3}$ for the root of $\frac{3}{7}$ within $\frac{1}{15}$.

By employing decimals in approximating the root of the numerator of the fraction $\frac{21}{49}$, we obtain 4,583 for the approximate root of the numerator 21, which is to be divided by the root of the new denominator. The quotient thence arising carried to three places of decimals becomes 0,655.

105. We are now prepared to resolve all equations involving only the second power of the unknown quantity connected with known quantities.

We have only to *collect into one member all the terms containing this power, to free it from the quantities, by which it is multiplied* (11); *we then obtain the value of the unknown quantity by extracting the square root of each member.*

Let there be, for example, the equation

$$\frac{5}{7}x^2 - 8 = 4 - \frac{2}{3}x^2.$$

Making the divisors to disappear, we find first

$$15x^2 - 168 = 84 - 14x^2.$$

Transposing to the first member the term $14x^2$, and to the second the term 168, we have

$$15x^2 + 14x^2 = 84 + 168,$$

or

$$29x^2 = 252$$

and

$$x^2 = \frac{252}{29},$$

$$x = \sqrt{\frac{252}{29}}$$

It should be carefully observed, that to denote the root of the fraction $\frac{252}{29}$, the sign $\sqrt{}$ is made to descend below the line, which separates the numerator from the denominator. If it were written thus $\frac{\sqrt{252}}{29}$, the expression would designate the quotient arising from the square root of the number 252 divided by 29; a result different from $\sqrt{\frac{252}{29}}$, which denotes, that the division is to be performed before the root is extracted.

Let there be the literal equation

$$ax^2 + b^3 = cx^2 + d^3;$$

proceeding as with the above, we obtain successively

$$ax^2 - cx^2 = d^3 - b^3$$

$$x^2 = \frac{d^3 - b^3}{a - c}$$

$$x = \sqrt{\frac{d^3 - b^3}{a - c}}.$$

I would remark here, that in order to designate the square root of a compound quantity, the upper line must be extended over the whole radical quantity.

The root of the quantity $4a^2b - 2b^3 + c^3$ is written thus,

$$\sqrt{4a^2b - 2b^3 + c^3},$$

or rather

$$\sqrt{}(4a^2b - 2b^3 + c^3),$$

by substituting, for the line extended over the radical quantity, a parenthesis including all the parts of the quantity, the root of which is required. This last expression may often appear preferable to the other (35).

In general, every equation of the second degree of the kind we are here considering, may, by a transposition of its terms, be reduced to the form

$$\frac{px^2}{q} = a,$$

$\frac{p}{q}$ designating the coefficient, whatever it may be, of x^2. We then obtain

$$x^2 = \frac{aq}{p}$$

$$x = \sqrt{\frac{aq}{p}}.$$

106. With respect to numbers taken independently, this solution is complete, since it is reduced to an operation upon the number either entire or fractional, which the quantity $\frac{aq}{p}$ represents, an arithmetical operation leading always to an exact result, or to one, which approaches the truth very nearly. But in regard to the signs, with which the quantities may be affected, there remains, after the square root is extracted, an ambiguity, in consequence of which every equation of the second degree admits of two solutions, while those of the first degree admit of only one.

Thus in the general equation $x^2 = 25$, the value of x, being the quantity, which, raised to its square, will produce 25, may, if we consider the quantities algebraically, be affected either with the sign $+$ or $-$; for whether we take $+5$, or -5 for this value, we have for the square

$$+5 \times +5 = +25, \quad \text{or} \quad -5 \times -5 = +25;$$

we may therefore take

$$x = +5,$$

or

$$x = -5.$$

For the same reason, from the general equation

$$x^2 = \frac{aq}{p},$$

we have

$$x = +\sqrt{\frac{aq}{p}},$$

or

$$x = -\sqrt{\frac{aq}{p}}.$$

Both these expressions are comprehended in the followin

$$x = \pm\sqrt{\frac{aq}{p}},$$

in which the double sign $\pm$ shows, that the numerical value

$$\sqrt{\frac{aq}{p}}$$

may be affected with the sign $+$ or $-$.

From what has been said, we deduce the general rule, *th double sign $\pm$ is to be considered as affecting the square r every quantity whatever.*

It may be here asked, why x, as it is the square root of *x* not also affected with the double sign $\pm$? We may answer that the letter x, having been taken without a sign, that is, the sign $+$, as the representative of the unknown quantity, it value when in this state, which is the subject of inquiry; that when we seek a number x, the square of which is b, fo ample, there can be only two possible solutions; $x = +$ $x = -\sqrt{b}$. Again, if in resolving the equation $x^2 = b$ write $\pm x = \pm\sqrt{b}$, and arrange these expressions in all th ferent ways, of which they are capable, namely;

$$+x = +\sqrt{b}, \quad -x = -\sqrt{b},$$
$$+x = -\sqrt{b}, \quad -x = +\sqrt{b},$$

we come to no new result, since by transposing all the ter the equations $-x = -\sqrt{b}$, $-x = +\sqrt{b}$, or, which is the thing, by changing all the signs (57), these equations be identical with the first.

107. It follows from the nature of the signs, that if the se member of the general equation

$$x^2 = \frac{aq}{p}$$

were a negative number, the equation would be absurd, since the square of a quantity affected either with the sign +, or —, having always the sign +, no quantity, the square of which is negative, can be found either among positive or negative quantities.

This is what is to be understood, when we say, that *the root of a negative quantity is imaginary.*

If we were to meet with the equation

$$x^2 + 25 = 9,$$

we might deduce from it

$$x^2 = 9 - 25$$

or

$$x^2 = \quad -16;$$

but there is no number, which, multiplied by itself, will produce — 16. It is true, that — 4 multiplied by + 4, gives — 16; but as these two quantities have different signs, they cannot be considered as equal, and consequently their product is not a square. This species of contradiction, which will be more fully considered hereafter, must be carefully distinguished from that, mentioned in art. 58, which disappears by simply changing the sign of the unknown quantity; here it is the sign of the square x^2, which is to be changed.

108. To be complete, an equation of the second degree, with only one unknown quantity, must have three kinds of terms, namely, those involving the square of the unknown quantity, others containing the unknown quantity of the first degree, lastly, such as comprehend only known quantities. The following equations are of this kind;

$$x^2 - 4x = 12, \qquad 4x - \tfrac{3}{5}x^2 = 4 - 2x.$$

The first is, in some respects, more simple than the second, because it contains only three terms, and the square of x is positive, and has only unity for a coefficient. It is to this last form, that we are always to reduce equations of the second degree before resolving them; they may then be represented by this formula,

$$x^2 + px = q,$$

in which p and q denote known quantities, either positive or negative.

It is evident, that we may reduce all equations of the second degree to this state, 1. by collecting into one member all the terms involving x (10), 2. by changing the sign of each term of the equation, in order to render that of x^2 positive, if it was before negative (57), 3. by dividing all the terms of the equation

by the multiplier of x^2, if this square have a multiplier (11), or by multiplying by its divisor, if it be divided by any number (12).

If we apply what has just been said to the equation

$$4x - \tfrac{3}{5}x^2 = 4 - 2x,$$

we have, by collecting into the first member all the terms involving x,

$$-\tfrac{3}{5}x^2 + 6x = 4,$$

by changing the signs

$$\tfrac{3}{5}x^2 - 6x = -4,$$

multiplying by the divisor 5,

$$3x^2 - 30x = -20,$$

dividing by the multiplier 3,

$$x^2 - 10x = -\tfrac{20}{3}.$$

If we now compare this equation with the general formula

$$x^2 + px = q,$$

we shall have

$$p = -10, \quad q = -\tfrac{20}{3}.$$

109. In order to arrive at the solution of equations thus prepared, we should keep in mind what has been already observed (34), namely, that the square of a quantity, composed of two terms, always contains the square of the first term, double the product of the first term multiplied by the second, and the square of the second; consequently the first member of the equation

$$x^2 + 2ax + a^2 = b,$$

in which a and b are known quantities, is a perfect square, arising from $x + a$, and may be expressed thus,

$$(x + a)(x + a) = b.$$

If we take the square root of the first member and indicate that of the second, we have

$$x + a = \pm\sqrt{b},$$

an equation which, considered with respect to x, is only of the first degree; and from which we obtain, by transposition,

$$x = -a \pm \sqrt{b}.$$

An equation of the second degree may therefore be easily resolved, whenever it can be reduced to the form

$$x^2 + 2ax + a^2 = b,$$

that is, whenever its first member is a perfect square.

But the first member of the general equation

$$x^2 + px = q,$$

contains already two terms, which may be considered as form-

ing part of the square of a binomial; namely, x^2 which is the square of the first term x, and px, or double the first multiplied by the second, which second is consequently only half of p, or $\frac{1}{2}p$. To complete the square of the binomial $x+\frac{1}{2}p$, there must be also the square of the second term $\frac{1}{2}p$; but this square may be formed, since p and $\frac{1}{2}p$ are known quantities, and it may be added to the first member, if, to preserve the equality of the two members, it be added at the same time to the second; and this last member will still be a known quantity.

As the square of $\frac{1}{2}p$ is $\frac{1}{4}p^2$, if we add it to the two members of the proposed equation,

$$x^2+px=q,$$

we shall have

$$x^2+px+\tfrac{1}{4}p^2=q+\tfrac{1}{4}p^2.$$

The first member of this result is the square of $x+\frac{1}{2}p$; taking then the root of the two members, we have

$$x+\tfrac{1}{2}p=\pm\sqrt{q+\tfrac{1}{4}p^2} \quad (106);$$

by transposition this becomes

$$x=-\tfrac{1}{2}p\pm\sqrt{q+\tfrac{1}{4}p^2},$$

or which is the same thing

$$x=-\tfrac{1}{2}p+\sqrt{q+\tfrac{1}{4}p^2}$$

and

$$x=-\tfrac{1}{2}p-\sqrt{q+\tfrac{1}{4}p^2}.$$

I have prefixed the sign $+$ to the second term $\frac{1}{2}p$, of the root of the first member of the above equation, because the second term of this member is positive; the sign $-$ is to be prefixed in the contrary case, because the square $x^2-2ax+a^2$ answers to the binomial $x-a$.

Any equation whatever of the second degree may be resolved, by referring it to the general formula

$$x^2+px=q;$$

or more expeditiously, by performing immediately upon the equation the operations represented under this formula, which, expressed in general terms, are as follows;

To make the first member of the proposed equation a perfect square, by adding to it, and also to the second, the square of half the given quantity, by which the first power of the unknown quantity is multiplied; then to extract the square root of each member, observing, that the root of the first member is composed of the unknown

quantity, and half of the given number, by which the unknown quantity in the second term is multiplied, taken with the sign of this quantity, and that the root of the second member must have the double sign $\pm$, *and be indicated by the sign* $\sqrt{}$, *if it cannot be obtained directly.*

See this illustrated by examples.

110. *To find a number such, that if it be multiplied by* 7, *and this product be added to its square, the sum will be* 44.

The number sought, being represented by x, the equation will evidently be

$$x^2 + 7x = 44.$$

In order to resolve this equation, we take $\frac{7}{2}$, half of the coefficient 7, by which x is multiplied; raising it to its square we obtain $\frac{49}{4}$; this added to each member gives

$$x^2 + 7x + \tfrac{49}{4} = 44 + \tfrac{49}{4};$$

reducing the second member to a single term, we have

$$x^2 + 7x + \tfrac{49}{4} = \tfrac{225}{4}.$$

The root of the first member, according to the rule given above, is $x + \frac{7}{2}$, and we find for that of the second $\frac{15}{2}$; whence arises the equation

$$x + \tfrac{7}{2} = \pm \tfrac{15}{2},$$

from which we obtain

$$x = -\tfrac{7}{2} \pm \tfrac{15}{2},$$

or

$$x = -\tfrac{7}{2} + \tfrac{15}{2} = \tfrac{8}{2} = 4,$$

$$x = -\tfrac{7}{2} - \tfrac{15}{2} = -\tfrac{22}{2} = -11.$$

The first value of x solves the question in the sense, in which it was enunciated, since we have by this value

$$\begin{array}{lr} & x^2 = 16 \\ & 7x = 28 \\ \hline \text{sum} & 44. \end{array}$$

As to the second value of x, since it is affected with the sign $-$, the term $7x$, which becomes

$$7 \times -11 = -77,$$

must be subtracted from x^2, so that the enunciation of the question resolved by the number 11 is this,

To find a number such, that 7 *times this number being subtracted from its square, the remainder will be* 44.

The negative value then here modifies the question in a manner, analogous to what takes place, as we have already seen, in equations of the first degree.

If we put the question, as enunciated above, into an equation, we obtain

$$x^2 - 7x = 44,$$

this becomes, when resolved,

$$x^2 - 7x + \tfrac{49}{4} = 44 + \tfrac{49}{4},$$
$$x^2 - 7x + \tfrac{49}{4} = \tfrac{225}{4},$$
$$x - \tfrac{7}{2} = \pm \tfrac{15}{2},$$
$$x = \tfrac{7}{2} \pm \tfrac{15}{2},$$
$$x = \tfrac{22}{2} = 11,$$
$$x = \tfrac{7}{2} - \tfrac{15}{2} = -\tfrac{8}{2} = -4.$$

The negative value of x becomes positive, as it satisfies precisely the new enunciation, and the positive value, which does not thus satisfy it, becomes negative.

Hence we see, that in equations of the second degree, algebra unites under the same formula two questions, which have a certain analogy to each other.

111. Sometimes enunciations, which produce equations of the second degree, admit of two solutions. The following is an example;

To find a number such, that if 15 *be added to its square, the sum will be equal to* 8 *times this number.*

Let x be the number sought; the equation arising from the problem is then

$$x^2 + 15 = 8x.$$

This equation reduced to the form prescribed in art. 108, becomes

$$x^2 - 8x = -15,$$
$$x^2 - 8x + 16 = -15 + 16,$$
$$x^2 - 8x + 16 = 1,$$
$$x - 4 = \pm 1,$$
$$x = 4 \pm 1,$$

or

$$x = 5,$$
$$x = 3.$$

There are therefore two different numbers 5 and 3, which fulfil the conditions of the question.

112. Questions sometimes occur, which cannot be resolved precisely in the sense of the enunciation, and which require to be modified. This is the case, when the two roots of the equation are negative, as in the following example,

$$x^2 + 5x + 6 = 2.$$

This equation, which denotes, that the *square of the number sought, augmented by 5 times this number, and also by 6, will give a sum equal to 2,* evidently cannot be verified by addition, as is implied, since 6 already exceeds 2. Indeed if we resolve it, we find successively

$$x^2 + 5x = -4$$
$$x^2 + 5x + \tfrac{25}{4} = \tfrac{25}{4} - 4 = \tfrac{9}{4}$$
$$x + \tfrac{5}{2} = \pm \tfrac{3}{2}$$
$$x = -\tfrac{5}{2} + \tfrac{3}{2} = -1.$$
$$x = -\tfrac{5}{2} - \tfrac{3}{2} = -4.$$

From the sign $-$, with which the numbers 1 and 4 are affected, it may be seen, that the term $5x$ must be subtracted from the others, and that the true enunciation for both values is,

To find a number such, that if 5 times this number be subtracted from its square, and 6 be added to the remainder, the result will be 2.

This enunciation furnishes the equation

$$x^2 - 5x + 6 = 2,$$

which gives for x the two positive values 1 and 4.

113. Again, let the following problem be proposed;

To divide a number p *into two parts, the product of which shall be equal to* q.

If we designate one of these parts by x, the other will be expressed by $p - x$, and their product will be $px - x^2$; we have then the equation

$$px - x^2 = q,$$

or, changing the signs,

$$x^2 - px = -q;$$

resolving this last, we find

$$x = \tfrac{1}{2}p \pm \sqrt{\tfrac{1}{4}p^2 - q}.$$

If now we suppose

$$p = 10, \qquad q = 21,$$

we have

$$x = 5 \pm \sqrt{25 - 21},$$

or

$$x = 5 \pm 2,$$
$$x = 7,$$
$$x = 3,$$

that is, one of the parts will be 7, and the other consequently $10 - 7$, or 3.

If on the contrary we take 3 for x, the other part will be

10 — 3 or 7; so that the enunciation, as it stands, admits, strictly speaking, of only one solution, since the second amounts simply to a change in the order of the parts.

If we examine carefully the value of x in the question we have been considering, we shall see that we cannot take any numbers indifferently for p and q, for if q exceed $\frac{p^2}{4}$, or the square of $\frac{1}{2}p$, the quantity $\frac{p^2}{4} - q$ becomes negative, and we are presented with that species of absurdity mentioned in art. 107.

If we take, for example,

$$p = 10 \text{ and } q = 30,$$

we have

$$x = 5 \pm \sqrt{25 - 30} = 5 \pm \sqrt{-5};$$

the problem then with these assumptions is impossible.

114. The absurdity of questions, which lead to imaginary roots, is discovered only by the result, and we may wish to determine by characters, which are found nearer to the enunciation, in what consists the absurdity of the problem, which gives rise to that of the solution; this we shall be enabled to do by the following consideration.

Let d be the difference of the two parts of the proposed number; the greater part will be $\frac{p}{2} + \frac{d}{2}$, the less $\frac{p}{2} - \frac{d}{2}$ (3); but it has been proved (29, 30 & 34) that

$$\left(\frac{p}{2} + \frac{d}{2}\right)\left(\frac{p}{2} - \frac{d}{2}\right) = \frac{p^2}{4} - \frac{d^2}{4};$$

therefore the product of the two parts of the proposed number, whatever they may be, will always be less than $\frac{p^2}{4}$, or than the square of half their sum, so long as d is any thing but zero; when d is nothing, each of the two parts being equal to $\frac{p}{2}$, their product will be only $\frac{p^2}{4}$. It is then absurd to require it to be greater; and it is just, that algebra should answer in a manner contradictory to established principles, and thereby show, that what is sought does not exist.

What has been proved concerning the equation

$$x^2 - px = -q,$$

furnished by the preceding question, is true of all those of the

second degree, where q is negative in the second member, and consequently equations, which produce imaginary roots, since the term placed under the radical sign, preserves always the sign, whatever may be that of p. Indeed it is evident that the equation

$$x^2 + p x = - q, \text{ or } x^2 + p x + q = 0,$$

will admit of no positive solution, since the first member contains only affirmative terms; and to ascertain whether the unknown quantity x can be negative, we have only to change x into $-y$. The unknown quantity y would then have positive values, which would be furnished by the equation

$$y^2 - p y + q = 0, \text{ or } y^2 - p y = - q,$$

which is precisely the same as that in the preceding article; but as the values of x can be real, only when those of y would be so, they become therefore imaginary in the case under consideration, when q exceeds $\frac{p^2}{4}$.

It will be perceived then from what has been said, how, and for what reason, *when the known term of an equation of the second degree is negative in the second member, and greater than the square of half the coefficient of the first power of the unknown quantity, the equation can have only imaginary roots.*

115. The expressions

$$\sqrt{-b}, \quad a + \sqrt{-b},$$

and in general those, which involve the square root of a negative quantity, are called *imaginary quantities.** They are mere symbols of absurdity, that take the place of the value, which we should have obtained, if the question had been possible.

They are not however to be neglected in the calculation, because it sometimes happens, that when they are combined according to certain laws, the absurdity disappears, and the result becomes real. Examples of this kind will be found in the *Supplement* to this treatise.

116. As it is important, that learners should have just ideas respecting all those analytical *facts*, which appear to be derived from familiar notions, I have thought it proper to add some observations to what has been said (106), on the necessity of admitting two solutions in equations of the second degree.

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* It would be more correct to say, *imaginary expressions* or *symbols* as they are not quantities.

I shall show that, *if there exists a quantity* a, *which, substituted in the place of* x, *verifies the equation of the second degree* $x^2 + px = q$, *and is consequently the value of* x, *this unknown quantity will still have another value.* Now, if we substitute a for x, the result will be $a^2 + pa = q$; and since, by supposition, a represents the value of x, q will be necessarily equal to the quantity $a^2 + pa$; we may then write this quantity in the place of q, in the proposed equation, which thus becomes

$$x^2 + px = a^2 + pa.$$

Transposing all the terms of the second member, we have

$$x^2 + px - a^2 - pa = 0,$$

which may be written,

$$x^2 - a^2 + p(x - a) = 0;$$

and because

$$x^2 - a^2 = (x + a)(x - a), \quad (34),$$

it is obvious, at once, that the first member is divisible by $x - a$, and will give an exact quotient, namely, $x + a + p$; we have then,

$$x^2 + px - q = x^2 - a^2 + p(x - a) = (x - a)(x + a + p).$$

Now it is evident, that a product is equal to zero, when any one of its factors whatever becomes nothing; we shall have then

$$(x - a)(x + a + p) = 0,$$

not only when $x - a = 0$, which gives

$$x = a,$$

but also when $x + a + p = 0$, from which is deduced

$$x = -a - p.$$

Therefore if a is one of the values of x, $-a-p$ will necessary be the other.

This result agrees with the two values comprehended in the formula

$$x = -\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2};$$

for if we take for a the first value, $-\frac{1}{2}p + \sqrt{q + \frac{1}{4}p^2}$, we obtain for the other

$$-a - p = +\tfrac{1}{2}p - \sqrt{q + \tfrac{1}{4}p^2} - p = -\tfrac{1}{2}p - \sqrt{q + \tfrac{1}{4}p^2},$$

which is in fact the second value.

These remarks contain the germ of the general theory of equations of whatever degree, as will appear hereafter, when the subject will be resumed.

117. The difficulty of putting a problem into an equation, is the same in questions involving the second and higher powers, as

in those involving only the first, and consists always in disentangling and expressing distinctly in algebraic characters all the conditions comprehended in the enunciation. The preceding questions present no difficulty of this sort; and although the learner is supposed to be well exercised in those of the first degree, I shall proceed to resolve a few questions, which will furnish occasion for some instructive remarks.

A person employed two labourers, allowing them different wages; the first received, at the end of a certain number of days, 96 *francs, and the second, having worked six days less, received only* 54 *francs; if this last had worked the whole number of days, and the other had lost six days, they would both have received the same sum; it is required to find how many days each worked, and what sum each received for a day's work.*

This problem, which at first view appears to contain several unknown quantities, may be easily solved by means of one, because the others may be readily expressed by this.

If x represent the number of days' work of the first labourer, $x-6$ will be the number of days' work of the second,

$\frac{96^{\text{fr.}}}{x}$ will be the daily wages of the first,

$\frac{54}{x-6}$ the daily wages of the second;

if this last had worked x days, he would have earned

$$x\times\frac{54}{x-6}\ \text{or}\ \frac{54x}{x-6},$$

and the first, working $x-6$ days, would have received only

$$(x-6)\frac{96}{x},\ \text{or}\ \frac{96(x-6)}{x}.$$

The equation of the problem then will be

$$\frac{54x}{x-6}=\frac{96(x-6)}{x}.$$

The first step is to make the denominators to disappear; the equation then becomes

$$54x^2=96(x-6)(x-6).$$

As the numbers 54 and 96 are both divisible by 6, the result may be simplified by division, we shall then have

$$9x^2=16(x-6)(x-6).$$

This last equation may be prepared for solution according to the rule given art. 108, but as the object of this rule is to enable us

with more facility to extract the root of each member of the equation proposed, it is here unnecessary, because the two members are already presented under the form of squares; for it is evident, that $9x^2$ is the square of $3x$, and $16(x-6)(x-6)$ the square of $4(x-6)$. We have then

$$3x = \pm 4(x-6),$$

from which may be deduced

$$3x = 4x - 24, \quad x = 24$$

$$3x = -4x + 24, \quad x = \tfrac{24}{7}.$$

By the first solution, the first labourer worked 24 days, and consequently earned $\frac{96}{24}$ or 4 francs per day, while the second worked only 18 days, and received $\frac{54}{18}$ or 3 francs per day.

The second solution answers to another numerical question, connected with the equation under consideration, in a manner analogous to what was noticed in art. 111.

118. *A banker receives two notes against the same person; the first of* 550 *francs, payable in seven months, the second of* 720 *francs, payable in four months, and gives for both the sum of* 1200 *francs; it is required to find, what is the annual rate of interest, according to which these notes are discounted.*

In order to avoid fractions in expressing the interest for seven months and four months, we shall represent by $12x$ the sum, which will amount in one year to 100 francs; the interest of one month will then be x. The present value of the first note will accordingly be found by the proportion,

$$100 + 7x : 100 :: 550 : \frac{55000}{100 + 7x} \quad (\textit{Arith.}\ 120);$$

and the present value of the second note by the proportion,

$$100 + 4x : 100 :: 720 : \frac{72000}{100 + 4x}.$$

By uniting these values, we obtain for the equation of the problem,

$$\frac{55000}{100 + 7x} + \frac{72000}{100 + 4x} = 1200.$$

Dividing each of the members by 200, we have

$$\frac{275}{100 + 7x} + \frac{360}{100 + 4x} = 6;$$

making the denominators to disappear, we find successively,

$$275(100 + 4x) + 360(100 + 7x) = 6(100 + 7x)(100 + 4x),$$

$$27500 + 1100x + 36000 + 2520x = 60000 + 6600x + 168x^2,$$

which may be reduced to

$$168\,x^2 + 2980\,x = 3500\,;$$

dividing by 2, we obtain

$$84\,x^2 + 1490\,x = 1750,$$

which gives

$$x^2 + \frac{1590}{84}x = \frac{1740}{84}.$$

Comparing this equation with the formula,

$$x^2 + p\,x = q,$$

we have

$$p = \frac{1490}{84}, \qquad q = \frac{1750}{84}\,;$$

and the expression

$$x = -\frac{1}{2}p \pm \sqrt{\frac{p^2}{4} + q}$$

becomes

$$x = -\frac{745}{84} \pm \sqrt{\frac{745 \,.\, 745}{84 \,.\, 84} + \frac{1750}{84}}.$$

Reducing the fractions, we have

$$\frac{745 \,.\, 745 + 1750 \,.\, 84}{84 \,.\, 84} = \frac{702025}{84 \,.\, 84}\,;$$

then since the denominator of this fraction is a perfect square, we have only to extract the square root of its numerator. If we stop at thousandths, we find 837,869, for the root of 702025; this, taken with the denominator 84, gives for the values of x

$$x = -\frac{745}{84} + \frac{837{,}869}{84} = \frac{92{,}869}{84}$$

$$x = -\frac{745}{84} - \frac{837{,}869}{84} = -\frac{1582{,}869}{84}.$$

The first of these values is the only one, which solves the question in the sense, in which it was enunciated. Dividing the denominator of this fraction by 12, we have (*Arith.* 54.)

$$12\,x = \frac{92{,}869}{7} = 13{,}267\,;$$

that is, the annual interest is at the rate of 13,27 nearly.

119. The following question deserves attention on account of the character, which the expression for the unknown quantity presents.

To divide a number into two parts, the squares of which shall be in a given ratio.

Let a be the given number,

m the ratio of the squares of its two parts,

x one of these parts;

he other will be $a-x$.

We shall then have, according to the enunciation,

$$\frac{x^2}{(a-x)(a-x)}=m.$$

This may be resolved in two ways; we may either reduce it o the form $x^2+px=q$, and then resolve it by the common nethod; or since the fraction

$$\frac{x^2}{(a-x)(a-x)}$$

s a square, the numerator and denominator being each a square, ve thence conclude at once

$$\frac{x}{a-x}=\pm\sqrt{m},$$

$$x=\pm(a-x)\sqrt{m}.$$

By resolving separately the two equations of the first degree omprehended in this formula, namely,

$$x=+(a-x)\sqrt{m}$$

$$x=-(a-x)\sqrt{m},$$

ve have

$$x=\frac{a\sqrt{m}}{1+\sqrt{m}}$$

$$x=\frac{-a\sqrt{m}}{1-\sqrt{m}}.$$

By the first solution, the second part of the number proposed is

$$a-\frac{a\sqrt{m}}{1+\sqrt{m}}=\frac{a+a\sqrt{m}-a\sqrt{m}}{1+\sqrt{m}}=\frac{a}{1+\sqrt{m}};$$

and the two parts

$$\frac{a\sqrt{m}}{1+\sqrt{m}} \text{ and } \frac{a}{1+\sqrt{m}}$$

are both, as the enunciation requires, less than the number proposed.

By the second solution we have

$$a+\frac{a\sqrt{m}}{1-\sqrt{m}}=\frac{a-a\sqrt{m}+a\sqrt{m}}{1-\sqrt{m}}=\frac{a}{1-\sqrt{m}};$$

and the two parts are

$$-\frac{a\sqrt{m}}{1-\sqrt{m}} \text{ and } \frac{a}{1-\sqrt{m}}.$$

Their signs being opposite, the number a is strictly no longer their sum, but their difference.

If we make $m = 1$, that is, if we suppose that the squares of the two parts sought are equal, we have

$$\sqrt{m} = 1;$$

and the first solution will give two equal parts,

$$\frac{a}{2}, \quad \frac{a}{2},$$

a conclusion, that is self-evident, while the second solution gives for the results two infinite quantities (68), namely,

$$\frac{-a}{1-1} \text{ or } \frac{-a}{0}, \quad \text{and } \frac{a}{1-1} \text{ or } \frac{a}{0}.$$

This is necessary, for it is only by considering two quantities infinitely great, with respect to their difference a, that we can suppose the ratio of their squares equal to unity.

Now, let there be the two quantities x and $x - a$, the ratio of their squares will be

$$\frac{x^2}{x^2 - 2ax + a^2};$$

dividing the two terms of this fraction by x^2, we obtain

$$\frac{1}{1 - \frac{2a}{x} + \frac{a^2}{x^2}};$$

but it is evident, that the greater the number x, the less will be the fractions $\frac{2a}{x}$, $\frac{a^2}{x^2}$, and the more nearly will the above ratio approach to $\frac{1}{1}$, or 1.

120. Now in order to compare the general method with that, which we have just employed, we develop the equation

$$\frac{x^2}{(a-x)(a-x)} = m;$$

and we have successively

$$x^2 = m(a-x)(a-x$$
$$x^2 = a^2 m - 2amx + mx^2$$
$$x^2 - mx^2 + 2amx = a^2 m$$
$$(1-m)x^2 + 2amx = a^2 m$$
$$x^2 + \frac{2amx}{1-m} = \frac{a^2 m}{1-m},$$

making $$p = \frac{2am}{1-m}, \qquad q = \frac{a^2 m}{1-m},$$

the general formula gives

$$x=-\frac{a\,m}{1-m}\pm\sqrt{\frac{a^2\,m^2}{(1-m)(1-m)}+\frac{a^2\,m}{1-m}}.$$

These values of x appear very different from those, which were found above; yet they may be reduced to the same; and in this consists the utility of the example, on which I am employed. It will serve to show the importance of those transformations, which different algebraic operations produce in the expression of quantities.

We must first reduce the two fractions comprehended under the radical sign to a common denominator. This may be done by multiplying the two terms of the second by $1-m$, we have then

$$\frac{a^2\,m^2}{(1-m)(1-m)}+\frac{a^2\,m}{1-m}=\frac{a^2\,m^2+a^2\,m\,(1-m)}{(1-m)(1-m)}=$$

$$\frac{a^2\,m^2+a^2\,m-a^2m^2}{(1-m)(1-m)}=\frac{a^2\,m}{(1-m)(1-m)}.$$

The denominator being a square, it is only necessary to extract the root of the numerator, we then have

$$\sqrt{\frac{a^2\,m^2}{(1-m)(1-m)}+\frac{a^2\,m}{1-m}}=\frac{\sqrt{a^2\,m}}{1-m};$$

but the expression $\sqrt{a^2\,m}$ may be further simplified.

It is evident that the square of a product is composed of the product of the squares of each of its factors, for example,

$$b\,c\,d\times b\,c\,d=b^2\,c^2\,d^2,$$

and consequently the root of $b^2\,c^2\,d^2$ is simply the product of the roots b, c and d, of the factors b^2, c^2 and d^2. Applying this principle to the product $a^2\,m$, we see that its root is the product of a, the root of a^2, by $\sqrt{m}$, which denotes the root of m, or that

$$\sqrt{a^2\,m}=a\sqrt{m}.$$

It follows from these different transformations, that

$$x=-\frac{a\,m}{1-m}\pm\frac{a\sqrt{m}}{1-m},$$

or

$$x=-\frac{a\,m-a\sqrt{m}}{1-m}$$

$$x=-\frac{a\,m+a\sqrt{m}}{1-m}.$$

These expressions, however simple, are still not the same as those given in the preceding article; if, moreover, we seek to verify them for the case, in which $m=1$, they become

17

$$x=\frac{-a+a}{1-1}=\frac{0}{0}$$

$$x=\frac{-a-a}{1-1}=\frac{-2a}{0}.$$

We find, in the second, the symbol of infinity, as in the preceding article, but the first presents this indeterminate form, $\frac{0}{0}$, of which we have already seen examples in articles 69 and 70; and before we pronounce upon its value, it is proper to examine, whether it does not belong to the case stated in art. 70; whether there is not some factor common to the numerator and denominator which the supposition of $m=1$ renders equal to zero.

The expression $\qquad \dfrac{-am+a\sqrt{m}}{1-m}$

may be resolved into

$$\frac{a(-m+\sqrt{m})}{1-m}=\frac{a(\sqrt{m}-m)}{1-m}.$$

It is here evident, that the numerator does not become 0 except by means of the factor $\sqrt{m}-m$; we must therefore examine, whether this last has not some factor in common with the denominator $1-m$. In order to avoid the inconvenience, arising from the use of the radical sign, let us make $\sqrt{m}=n$, then taking the squares, we have $m=n^2$; the quantities therefore

$$\sqrt{m}-m \text{ and } 1-m$$

become $\qquad n-n^2$ and $1-n^2$,

but $n-n^2=n(1-n)$, and $1-n^2=(1-n)(1+n)$ (34); restoring to the place of n its value $\sqrt{m}$, we have

$$\sqrt{m}-m=(1-\sqrt{m})\sqrt{m}$$

$$1-m=(1-\sqrt{m})(1+\sqrt{m}),$$

and consequently

$$\frac{a(\sqrt{m}-m)}{1-m}=\frac{a(1-\sqrt{m})\sqrt{m}}{(1-\sqrt{m})(1+\sqrt{m})}=\frac{a\sqrt{m}}{1+\sqrt{m}},$$

a result the same, as that found in art. 119.

In the same manner we may reduce the second value of x, observing that

$$\frac{-a\sqrt{m}-am}{1-m}=\frac{-a(1+\sqrt{m})\sqrt{m}}{(1-\sqrt{m})(1+\sqrt{m})}=\frac{-a\sqrt{m}}{1-\sqrt{m}},$$

as in art. 119.*

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* The example, which I have given at some length, corresponds with a problem resolved by Clairaut, in his Algebra, the enunciation of

It will be seen without difficulty, that we might have avoided radical expressions in the preceding calculations, by taking n^2 to represent the ratio, which the squares of the two parts of the proposed number have to each other; m would then have been the square root, which may always be considered as known, when the square is known; but we could not have perceived from the beginning the object of such a change in a given term, of which algebraists often avail themselves, in order to render calculations more simple. It is recommended to the learner therefore, to go over the solution again, putting m^2 in the place of m.

Of the extraction of the square root of algebraic quantities.

121. We have sufficiently illustrated, by the preceding example, the manner of conducting the solution of literal questions. We have given also an instance of a transformation, namely, that of $\sqrt{a^2 m}$ into $a\sqrt{m}$, which is worthy of particular attention; since by means of it, we have been able to reduce the factors, contained under a radical sign, to the smallest number possible, and thus to simplify very much the extraction of the remaining part of the root.

This transformation consists in *taking the roots of all the factors which are squares, and writing them without the radical sign, as multipliers of the radical quantity, and retaining under the radical sign all those factors, which are not squares.*

This rule supposes, that the student is already able to determine, whether an algebraic quantity is a square, and is acquainted with the method of extracting the root of such a quantity. In order to this, it is necessary to distinguish simple quantities from polynomials.

122. It is evident, from the rule given for the exponents in

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which is as follows; *To find on the line, which joins any two luminous bodies, the point where these two bodies shine with equal light.* I have divested this problem of the physical circumstances, which are foreign to the object of this work, and which only divert the attention from the character of the algebraic expressions. These expressions are very remarkable in themselves, and for this reason I have developed them more fully, than they were done in the work referred to.

multiplication, that *the second power of any quantity has an exponent double that of this quantity.*

We have, for example,

$$a^1 \times a^1 = a^2, \quad a^2 \times a^2 = a^4, \quad a^3 \times a^3 = a^6, \text{ \&c.}$$

It follows then, that *every factor, which is a square, must have an exponent which is an even quantity, and that the root of this factor is found by writing its letter with an exponent equal to half the original exponent.*

Thus we have

$$\sqrt{a^2} = a^1 \text{ or } a, \quad \sqrt{a^4} = a^2, \quad \sqrt{a^6} = a^3, \quad \text{\&c.}$$

With respect to numerical factors, their roots are extracted, when they admit of any, by the rules already given.

Whence the factors a^6, b^4, c^2, in the expression

$$\sqrt{64\, a^6\, b^4\, c^2},$$

are squares, and the number 64 is the square of 8; therefore *as the expression proposed is the product of factors, which are squares, it will have for a root the product of the roots of these several factors* (121); and consequently

$$\sqrt{64\, a^6\, b^4\, c^2} = 8\, a^3\, b^2\, c.$$

123. In other cases different from the above, *we must endeavour to resolve the proposed quantity, considered as a product, into two other products, one of which shall contain only such factors as are squares, and the other those factors, which are not squares.* To effect this, we must consider each of the quantities separately.

Let there be, for example,

$$\sqrt{72\, a^4\, b^3\, c^5}.$$

We see that among the divisors of 72, the following are perfect squares, namely, 4, 9 and 36; if we take the greatest, we have

$$72 = 36 \times 2.$$

As the factor a^4 is a square, we separate it from the others; passing then to the factor b^3, which is not a square, since 3 is an odd number, we observe that this factor may be resolved into two others, b^2 and b, the first of which is a square; we have then

$$b^3 = b^2 \,.\, b;$$

it is obvious also that

$$c^5 = c^4 \,.\, c.$$

By proceeding in the same manner with every letter, whose exponent is an odd number, the quantity is resolved thus,

$$72\, a^4\, b^3\, c^5 = 36 \,.\, 2\, a^4\, b^2 \,.\, b\, c^4 \,.\, c;$$

by collecting the factors, which are squares, it becomes

$$36\,a^4\,b^2\,c^4 \times 2\,b\,c.$$

Lastly, taking the root of the first product and indicating that of the second, we have

$$\sqrt{72\,a^4\,b^3\,c^5} = 6\,a^2\,b\,c^2\,\sqrt{2\,b\,c}.$$

See some examples of this kind of reduction with the steps, by which they are performed;

$$\sqrt{\frac{a^3}{b}} = \sqrt{a^2\,\frac{a}{b}} = a\sqrt{\frac{a}{b}} = a\sqrt{\frac{a\,b}{b^2}} = \frac{a}{b}\sqrt{a\,b};$$

$$6\sqrt{\frac{75}{98}a\,b^2} = 6\sqrt{\frac{25\;\;3\,a\,b^2}{49\,.\,2}} = 6\sqrt{\frac{25\,b^2\,.\,3\,a}{49\,.\,2}} =$$

$$\frac{6\,.\,5}{7}\,b\sqrt{\frac{3\,a}{2}} = \frac{30\,b}{7}\sqrt{\frac{3\,a}{2}};$$

$$\sqrt{\frac{a^2\,m^2}{n^2} + \frac{a^2\,m}{n}} = \sqrt{\frac{a^2\,m^2 + a^2\,m\,n}{n^2}} =$$

$$\sqrt{\frac{a^2}{n^2}\,(m^2 + m\,n)} = \frac{a}{n}\sqrt{m^2 + m\,n}.$$

It will be seen by the first of these examples, that the denominator of an algebraic fraction may be taken from under the radical sign by being made a complete square, in the same manner as we reduce the root of a numerical fraction (104.)

124. We now proceed to the extraction of the square root of polynomials. It must here be recollected, that no binomial is a perfect square, because every simply quantity raised to a square produces only a simple quantity, and the square of a binomial always contains three parts (34.)

It would be a great mistake to suppose the binomial $a+b$ to be the square root of a^2+b^2, although taken separately, a is the root of a^2, and b that of b^2; for the square of $a+b$, or $a^2+2\,a\,b+b^2$, contains the term $+\,2\,a\,b$, which is not found in the expression $\sqrt{a^2+b^2}$.

Let there be the trinomial

$$24\,a^2\,b^3\,c + 16\,a^4\,c^2 + 9\,b^6.$$

In order to obtain from this expression the three parts, which compose the square of a binomial, we must arrange it with reference to one of its letters, the letter a, for example; it then becomes

$$16\,a^4\,c^2 + 24\,a^2\,b^3\,c + 9\,b^6.$$

Now, whatever be the square root sought, if we suppose it ar-

ranged with reference to the same letter a, the square of its first term must necessarily form the first term, $16\,a^4\,c^2$, of the proposed quantity; double the product of the first term of the root by the second must give the second term, $24\,a^2\,b^3\,c$, of the proposed quantity; and the square of the last term of the root must give exactly the last term, $9\,b^6$, of the proposed quantity. The operation may be exhibited, as follows;

$$\begin{array}{r|l} 16\,a^4\,c^2 + 24\,a^2\,b^3\,c + 9\,b^6 & 4\,a^2\,c + 3\,b^3 \text{ root} \\ \underline{-16\,a^4\,c^2 \qquad\qquad\qquad\qquad} & 8\,a^2\,c + 3\,b^3 \\ +24\,a^2\,b^3\,c + 9\,b^6 & \\ \underline{-24\,a^2\,b^3\,c - 9\,b^6} & \\ 0 \qquad\quad 0 & \end{array}$$

We begin by finding the square root of the first term, $16\,a^4\,c^2$, and the result $4\,a^2\,c$ (122) is the first term of the root, which is to be written on the right, upon the same line with the quantity, whose root is to be extracted.

We subtract from the proposed quantity, the square, $16\,a^4\,c^2$ of the first term, $4\,a^2\,c$, of the root; there remain then only the two terms $24\,a^2\,b^3\,c + 9\,b^6$.

As the term $24\,a^2\,b^3\,c$ is double the product of the first term of the root, $4\,a^2\,c$, by the second, we obtain this last, by dividing $24\,a^2\,b^3\,c$ by $8\,a^2\,c$, double of $4\,a^2\,c$, which is written below the root; the quotient $3\,b^3$ is the second term of the root.

The root is now determined; and, if it be exact, the square of the second term will be $9\,b^6$, or rather, double of the first term of the root $8\,a^2\,c$ together with the second $3\,b^3$, multiplied by the second, will reproduce the two last terms of the square (91); therefore we write $+3\,b^3$ by the side of $8\,a^2\,c$, and multiply $8\,a^2\,c + 3\,b^3$ by $3\,b^3$; after the product is subtracted from the two last terms of the quantity proposed, nothing remains; and we conclude, that this quantity is the square of $4\,a^2\,c + 3\,b^3$.

It is evident that the same reasoning and the same process may be applied to all quantities composed of three terms.

125. When the quantity, whose root is to be extracted, has more than three terms, it is no longer the square of a binomial; but if we suppose it the square of a trinomial, $m + n + p$, and represent by l the sum $m + n$, this trinomial becoming now $l + p$, its square will be

$$l^2 + 2\,l\,p + p^2,$$

in which the square l^2 of the binomial $m+n$, produces, when developed, the terms $m^2+2\,m\,n+n^2$. Now, after we have arranged the proposed quantity, the first term will evidently be the square of the first term of the root, and the second will contain double the product of the first term of the root by the second of this root; we shall then obtain this last by dividing the second term of the proposed quantity by double the root of the first. Knowing then the two first terms of the root sought, we complete the square of these two terms, represented here by l^2; subtracting this square from the proposed quantity, we have for a remainder

$$2\,l\,p+p^2,$$

a quantity, which contains double the product of l, or of the first binomial $m+n$, by the remainder of the root, plus the square of this remainder. It is evident therefore that we must proceed with this binomial as we have done with the first term m of the root.

Let there be, for example, the quantity

$$64\,a^2\,b\,c+25\,a^2\,b^2-40\,a^3\,b+16\,a^4+64\,b^2\,c^2-80\,a\,b^2\,c;$$

we arrange it with reference to the letter a, and make the same disposition of the several parts of the operation as in the above example.

$$\begin{array}{llllll|l}
 & 16a^4 & -40a^3b & +25a^2b^2 & -80ab^2c & +64b^2c^2 & 4a^2-5ab+8bc \\
 & & & +64a^2bc & & & \overline{8a^2-5ab} \\
 & -16a^4 & & & & & 8a^2-10ab+8bc \\
\hline
\text{1st rem.} & & -40a^3b & +25a^2b^2 & -80ab^2c & +64b^2c^2 & \\
 & & & +64a^2bc & & & \\
 & & +40a^3b & -25a^2b^2 & & & \\
\hline
\text{2d. rem} \ldots\ldots & & & +64a^2bc & -80ab^2c & +64b^2c^2 & \\
 & & & -64a^2bc & +80ab^2c & -64b^2c^2 & \\
\hline
 & & & 0 & 0 & 0 &
\end{array}$$

We extract the square root of the first term $16\,a^4$, and obtain $4\,a^2$ for the first term of the root sought, the square of which is to be subtracted from the proposed quantity.

We double the first term of the root, and write the result $8\,a^2$ under the root; dividing by this the term $-40\,a^3\,b$, which begins the first remainder, we have $-5\,a\,b$ for the second term of the root; this is to be placed by the side of $8\,a^2$, we then multiply the whole by this second term, and subtract the result from the remainder, upon which we are employed.

Thus we have subtracted from the proposed quantity the square of the binomial $4a^2 - 5ab$; the second remainder can contain only double the product of this binomial, by the third term of the root, together with the square of this term; we take then double the quantity $4a^2 - 5ab$, or

$$8a^2 - 10ab,$$

which is written under $8a^2 - 5ab$, and constitutes the divisor to be used with the second remainder; the first term of the quotient, which is $8bc$, is the third of the root.

This term we write by the side of $8a^2 - 10ab$, and multiply the whole expression by it; the product being subtracted from the remainder under consideration, nothing is left; the quantity proposed therefore is the square of

$$4a^2 - 5ab + 8bc.$$

The above operation, which is perfectly analogous to that, which has been already applied to numbers, may be extended to any length we please.

Of the formation of powers and the extraction of their roots.

126. The arithmetical operation, upon which the resolution of equations of the second degree depends, and by which we ascend from the square of a quantity to the quantity, from which it is derived, or to the square root, is only a particular case of a more general problem, namely, *to find a number, any power of which is known.* The investigation of this problem leads to a result, that is still termed a root, the different kinds being called degrees, but the process is to be understood only by a careful examination of the steps by which a power is obtained, one operation being the reverse of the other, as we observe with respect to division and multiplication, with which it will soon be perceived that this subject has other relations.

It is by multiplication, that we arrive at the powers of entire numbers (24), and it is evident, that those of fractions also are formed by raising the numerator and denominator to the power proposed (96).

So also the root of a fraction, of whatever degree, is obtained by taking the corresponding root of the numerator and that of the denominator.

As algebraic symbols are of great use in expressing every thing, which relates to the composition and decomposition of

quantities, I shall first consider how the powers of algebraic expressions are formed, those of numbers being easily found by the methods that have already been given (24.)

Table of the first seven powers of numbers from 1 *to* 9.

1st	1	2	3	4	5	6	7	8	9
2d	1	4	9	16	25	36	49	64	81
3d	1	8	27	64	125	216	343	512	729
4th	1	16	81	256	625	1296	2401	4096	6561
5th	1	32	243	1024	3125	7776	16807	32768	59049
6th	1	64	729	4096	15625	46656	117649	262144	531441
7th	1	128	2187	16384	78125	279936	823543	2097152	4782969

This table is intended particularly to show with what rapidity the higher powers of numbers increase, a circumstance that will be found to be of great importance hereafter; we see, for instance, that the seventh power of 2 is 128, and that of 9 amounts to 4782969.

It will hence be readily perceived, that the powers of fractions, properly so called, decrease very rapidily, since the powers of the denominator become greater and greater in comparison with those of the numerator. The seventh power of $\frac{1}{2}$, for example, is $\frac{1}{128}$, and that of $\frac{1}{9}$ is only

$$\frac{1}{4782969}.$$

127. It is evident from what has been said, that in a product each letter has for an exponent the sum of the exponents of its several factors (26), that *the power of a simple quantity is obtained by multiplying the exponent of each factor by the exponent of this power.*

The third power of $a^2 b^3 c$, for example, is found by multiplying the exponents 2, 3, and 1, of the letters a, b and c, by 3, the exponent of the power required; we have then $a^6 b^9 c^3$; the operation may be thus represented,

$$a^2 b^3 c \times a^2 b^3 c \times a^2 b^3 c = a^{2\cdot 3} b^{3\cdot 3} c^{1\cdot 3}.$$

If the proposed quantity have a numerical coefficient, this coefficient must also be raised to the same power; thus the fourth power of $3 a b^2 c^5$, is

$$81 a^4 b^8 c^{20}.$$

128. With respect to the signs, with which simple quantities may be affected, it must be observed, that *every power, the exponent of which is an even number, has the sign +, and every power the exponent of which is an odd number, has the same sign as the quantity from which it is formed.*

In fact, powers of an even degree arise from the multiplication of an even number of factors; and the signs —, combined two and two in the multiplication, always give the sign + in the product (31). On the contrary, if the number of factors is uneven, the product will have the sign —, when the factors have this sign, since this product will arise from that of an even number of factors, multiplied by a negative factor.

129. In order to ascend from the power of a quantity, to the root from which it is derived, we have only to reverse the rules given above, that is, *to divide the exponent of each letter by that, which marks the degree of the root required.*

Thus we find the *cube root*, or *the root of the third degree*, of the expression $a^6\,b^9\,c^3$, by dividing the exponents 6, 9 and 3 by 3, which gives

$$a^2\,b^3\,c.$$

When the proposed expression has a numerical coefficient, its root must be taken for the coefficient of the literal quantity, obtained by the preceding rule.

If it were required, for example, to find the fourth root of $81\,a^4\,b^8\,c^{20}$, we see by referring to table art. 126, that 81 is the fourth power of 3; then dividing the exponent of each of the letters by 4, we obtain for the result

$$3\,a\,b^2\,c^5.$$

When the root of the numerical coefficient cannot be found by the table inserted above, it must be extracted by the methods to be given hereafter.

130. It is evident, that the roots of the literal part of simple quantities can be extracted, only when each of the exponents is divisible by that of the root; in the contrary case, we can only indicate the arithmetical operation, which is to be performed, whenever numbers are substituted in the place of the letters.

We use for this purpose the sign $\sqrt{\ }$; but to designate the degree of the root, we place the exponent as in the following expressions,

$$\sqrt[3]{a},\quad \sqrt[5]{a^2},$$

the first of which represents the cube root, or the root of the third degree of a, and the second the fifth root of a^2.

We may often simplify radical expressions of any degree whatever, by observing, according to art. 127, that *any power of a product is made up of the product of the same power of each of the factors*, and that consequently, *any root of a product is made up of the product of the roots of the same degree of the several factors.* It follows from this last principle, that, *if the quantity placed under the radical sign have factors, which are exact powers of the degree denoted by this sign, the roots of these factors may be taken separately, and their product multiplied by the root of the other factors indicated by the sign.*

Let there be, for example,

$$\sqrt[5]{96\, a^5\, b^7\, c^{11}}.$$

It may be seen that,

$$96 = 32 \times 3 = 2^5 \,.\, 3,$$

that $\quad a^5 \quad$ is the fifth power of a,

that $\quad b^7 = b^5 \,.\, b^2,$

that $\quad c^{11} = c^{10} \,.\, c;$

we have then

$$96\, a^5\, b^7\, c^{11} = 2^5\, a^5\, b^5\, c^{10} \times 3\, b^2\, c.$$

As the first factor $2^5\, a^5\, b^5\, c^{10}$, has for its fifth root the quantity $2\,a\,b\,c^2$, the expression becomes

$$\sqrt[5]{96\, a^5\, b^7\, c^{11}} = 2\,a\,b\,c^2 \sqrt[5]{3\, b^2\, c.}$$

131. As every even power has the sign + (128), a quantity, affected with the sign —, cannot be a power of a degree denoted by an even number, and it can have no root of this degree. It follows from this, that *every radical expression of a degree which is denoted by an even number, and which involves a negative quantity, is imaginary, thus*

$$\sqrt[4]{-a}, \quad \sqrt[6]{-a^4}, \quad b + \sqrt[8]{-a\,b^7},$$

are imaginary expressions.

We cannot therefore, either exactly or by approximation, assign for a degree, the exponent of which is an even number, any roots but those of positive quantities, and *these roots may be affected indifferently with the sign + or —*, because in either case, they will equally reproduce the proposed quantity with the sign +, and we do not know to which class they belong.

The same cannot be said of degrees expressed by an odd num-

ber, for here the powers have the same sign as their roots (128); and *we must give to the roots of these degrees the sign, with which the power is affected*; and no imaginary expressions occur.

132. It is proper to observe, that the application of the rule given in art. 129, for the extraction of the roots of simple quantities, by means of the exponent of their factors, leads to a more convenient method of indicating roots, which cannot be obtained algebraically, than by the sign $\sqrt{\ }$.

If it were required, for example, to find the third root of a^5, it is necessary, according to the rule given above, to divide the exponent 5 by 3; but as we cannot perform the division, we have for the quotient the fractional number $\frac{5}{3}$; and this form of the exponent indicates, that the extraction of the root is not possible in the actual state of the quantity proposed. We may therefore consider the two expressions

$$\sqrt[3]{a^5} \quad \text{and} \quad a^{\frac{5}{3}}$$

as equivalent.

The second however has this advantage over the first, that it leads directly to a more simple form, which the quantity $\sqrt[3]{a^5}$ is capable of assuming; for if we take the whole number contained in the fraction $\frac{5}{3}$, we have $1+\frac{2}{3}$ as an equivalent exponent; consequently,

$$a^{\frac{5}{3}} = a^{1+\frac{2}{3}} = a^1 \times a^{\frac{2}{3}} \text{ (25)};$$

from which it is evident, that the quantity $a^{\frac{5}{3}}$ is composed of two factors, the first of which is rational, and the other becomes $\sqrt[3]{a^2}$.

The same result indeed may be obtained from the quantity under the form $\sqrt[3]{a^5}$, by the rule given in art. 130, but the fractional exponent suggests it immediately. We shall have occasion to notice in other operations the advantages of fractional exponents.

We will merely observe for the present, that as the division of exponents, when it can be performed, answers to the extraction of roots, the indication of this division under the form of a fraction is to be regarded as the symbol of the same operation; whence,

$$\sqrt[n]{a^m} \quad \text{and} \quad a^{\frac{m}{n}}$$

are equivalent expressions.

We have rules then, which result from the assumed manner of expressing powers, which lead to particular symbols, as in art. 37, we arrived at the expression $a^0 = 1$.

133. It may be observed here, that as we divide one power by another by subtracting the exponent of the latter from that of the former (36), fractions of a particular description may readily be reduced to new forms.

By applying the rule above referred to we have

$$\frac{a^m}{a^n} = a^{m-n};$$

but if the exponent n of the denominator exceed the exponent m of the numerator, the exponent of the letter a in the second member will be negative.

If, for example, $m = 2$, $n = 3$, we have

$$\frac{a^2}{a^3} = a^{2-3} = a^{-1};$$

but by another method of simplifying the fraction $\frac{a^2}{a^3}$, we find it equal to $\frac{1}{a}$ the expressions

$$\frac{1}{a} \quad \text{and} \quad a^{-1}$$

are therefore equivalent.

In general, we obtain by the rule for the exponents

$$\frac{a^m}{a^{m+n}} = a^{m-m-n} = a^{-n},$$

and by another method

$$\frac{a^m}{a^{m+n}} = \frac{1}{a^n};$$

it follows from this, that the expressions

$$\frac{1}{a^n} \quad \text{and} \quad a^{-n}$$

are equivalent.

In fact, the sign $-$, which precedes the exponent n, being taken in the sense defined in art. 62, shows that the exponent in question arises from a fraction, the denominator of which contains the factor a, n times more than the numerator, which fraction is indeed $\frac{1}{a^n}$; we may therefore, in any case which occurs, substitute one of these expressions for the other.

The quantity $\frac{a^3 b^5}{c^2 d^6}$, for example, being considered as equivalent to

$$a^2 b^3 \times \frac{1}{c^2} \times \frac{1}{d^3},$$

may be reduced to the following form,

$$a^2 b^3 c^{-2} d^{-3};$$

that is, *we may transfer to the numerator all the factors of the denominator, by giving to their exponents the sign —.*

Reciprocally, *when a quantity contains factors, which have negative exponents, we may convert them into a denominator, observing merely to give to their exponents the sign +;* thus the quantity

$$a^2 b^3 c^{-2} d^{-3},$$

becomes

$$\frac{a^2 b^3}{c^2 d^3}.$$

Of the formation of the powers of compound quantities.

134. We shall begin this section by observing, that the powers of compound quantities are denoted by including these quantities in a parenthesis, to which is annexed the exponent of the power. The expression

$$(4a^2 - 2ab + 5b^2)^3,$$

for example, denotes the third power of the quantity,

$$4a^2 - 2ab + 5b^2.$$

This power may also be expressed thus

$$\overline{4a^2 - 2ab + 5b^2}^{\,3}.$$

135. Binomials next to simple quantities are the least complicated, yet if we undertake to form powers of these by successive multiplications, we in this way arrive only at particular results, as in art. 34, we obtained the second and third power; thus

$$(x+a)^2 = x^2 + 2ax + a^2$$
$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$
$$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$$
&c.

It is not easy from this table to fix upon the law, which determines the value of the numerical coefficients. But by considering how the terms are multiplied into each other, we perceive, that the coefficients have their origin in reductions depending on the equality of the factors, which form a power. This is rendered very evident by an arrangement, which prevents these reductions taking place.

It is sufficient for this purpose to give to the several binomials

to be multiplied different second terms. If we take, for example,

$$x+a,\quad x+b,\quad x+c,\quad x+d,\ \&c.$$

by performing the multiplications indicated below, and placing in the same column the terms, which involve the same power of x, we shall immediately find, that

$$\begin{aligned}
(x+a)(x+b) &= x^2 + ax + ab \\
& + bx \\
(x+a)(x+b)(x+c) &= x^3 + ax^2 + abx + abc \\
& + bx^2 + acx \\
& + cx^2 + bcx \\
(x+a)(x+b)(x+c)(x+d) &= x^4 + ax^3 + abx^2 + abcx + abcd \\
& + bx^3 + acx^2 + abdx \\
& + cx^3 + adx^2 + acdx \\
& + dx^3 + bcx^2 + bcdx \\
& + bdx^2 \\
& + cdx^2
\end{aligned}$$

Without carrying these products any further, we may discover the law according to which they are formed.

By supposing all the terms involving the same power of x, and placed in the same column, to form only one, as, for example,

$$ax^3 + bx^3 + cx^3 + dx^3 = (a+b+c+d)\,x^3,$$
&c.

1. *We find in each product one term more than there are units in the number of factors.*

2. *The exponent of* x *in the first term is the same as the number of factors, and goes on decreasing by unity in each of the following terms.*

3. *The greatest power of* x *has unity for its coefficient; the following, or that, whose exponent is one less, is multiplied by the sum of the second terms of the binomials; that, whose exponent is two less, is multiplied by the sum of the different products of the second terms of the binomials taken two and two; that whose exponent is three less, is multiplied by the sum of the different products of the second term of the binomials, taken three and three, and so on; in the last term, the exponent of* x, *being considered as zero* (37), *is equal to that of the first, diminished by as many units as there are factors employed, and this term contains the product of all the second terms of the binomials.*

It is manifest, that the form of these products must be subject to the same laws, whatever be the number of factors; as may be shown by other evidence beside that from analogy.

136. It will be seen immediately, that the products, of which we are speaking, must contain the successive powers of x, from that, whose exponent is equal to the number of factors employed, to that whose exponent is zero. To present this proposition under a general form, we shall express the number of factors by the letter m; the successive powers of x will then be denoted by

$$x^m, \quad x^{m-1}, \quad x^{m-2}, \quad \&c.$$

We shall employ the letters $A, \quad B, \quad C, \quad \ldots\ldots\ldots\ldots \; Y,$ to express the quantities, by which these powers, beginning with x^{m-1}, are to be multiplied; but as the number of terms, which depends on the particular value given to the exponent, will remain indeterminate, so long as this exponent has no particular value, we can write only the first and last terms of the expression, designating the intermediate terms by a series of points.

The formula then

$$x^m + A\,x^{m-1} + B\,x^{m-2} + C\,x^{m-3} \ldots\ldots + Y,$$

represents the product of any number m of factors,

$$x + a,\, x + b, \quad x + c,\, x + d, \; \&c.$$

If we multiply this by a new factor $x + l$, it becomes

$$\left.\begin{array}{l} x^{m+1} + A\,x^m + \;\; B\,x^{m-1} + \;\; C\,x^{m-2} \ldots\ldots \\ \quad\; + \; l\,x^m \; + l\,A\,x^{m-1} + l\,B\,x^{m-2} \ldots\ldots + l\,Y \end{array}\right\}.$$

It is evident, 1. that if A is the sum of the m second terms a, b, c, d, &c. $A + l$ will be that of the $m + 1$ second terms a, b, c, d, &c. l, and that consequently the expression employed to denote the coefficient will be true for the product of the degree $m + 1$, if it is true for that of the degree m.

2. If B is the sum of the products of the m quantities a, b, c, d, &c. taken two and two, $B + l\,A$, will express that of the products of the $m + 1$ quantities a, b, c, d, &c. l, taken also two and two; for A being the sum of the first, $l\,A$ will be that of their products by the new quantity introduced l; therefore the expression employed will be true for the degree $m + 1$, if it is for the degree m.

If C is the sum of the products of the m quantities a, b, c, d, &c. taken three and three, $C + l\,B$ will be that of the products of the $m + 1$ quantities a, b, c, d, &c. l, taken also three and three, since $l\,B$, from what has been said, will express the sum of the products of the first taken two and two, multiplied by the new quantity

introduced l; therefore the expression employed will be true for the degree $m+1$, if it is true for the degree m.

It will be seen, that this mode of reasoning may be extended to all the terms, and that the last $l\,Y$ will be the product of $m+1$ second terms.

The propositions laid down in art. 135, being true for expressions of the fourth degree, for example, will be so, according to what has just been proved, for those of the fifth, for those of the sixth; and being extended thus, from one degree to another, they may be shown to be true generally.

It follows from this, that the product of any number whatever m of binomial factors $x+a$, $x+b$, $x+c$, $x+d$, &c. being represented by

$$x^m + A\,x^{m-1} + B\,x^{m-2} + C\,x^{m-3} + \text{\&c.}$$

A will always be the sum of the m letters a, b, c, &c. B that of the products of these quantities, taken two and two, C that of the products of these quantities, taken three and three, and so on.

To comprehend the law of this expression in a single term, I take one, whose place is indeterminate, and which may be represented by $N\,x^{m-n}$.

This term will be the second, if we make $n=1$, the third, if we make $n=2$, the eleventh, if we make $n=10$, &c. In the first case, the letter N will be the sum of the m letters a, b, c, &c. in the second, that of their products, when taken two and two; in the third, that of their products, when taken ten and ten; and in general, that of their products, taken n and n.

137. To change the products

$$(x+a)\,(x+b),\ (x+a)\,(x+b)\,(x+c),$$
$$(x+a)\,(x+b)\,(x+c)\,(x+d),\ \text{\&c.}$$

into powers of $x+a$, namely, into

$$(x+a)^2, \qquad (x+a)^3,$$
$$(x+a)^4, \qquad \text{\&c.}$$

it is only necessary to make, in the development of these products,

$$a=b, \qquad a=b=c,$$
$$a=b=c=d, \qquad \text{\&c.}$$

All the quantities, by which the same power of x is multiplied, become in this case equal; thus the coefficient of the second term, which in the product

$$(x+a)\,(x+b)\,(x+c)\,(x+d) \text{ is } a+b+c+d,$$

is changed into $4a$; that of the third term in the same product, which is,

$$ab + ac + ad + bc + bd + cd,$$

becomes $6a^2$. Hence it is easy to see, that the coefficients of the different powers of x, will be changed into a single power of a, repeated as many times as there are terms, and distinguished by the number of factors contained in each of these terms. Thus, the coefficient N, by which the power x^{m-n} is multiplied, will, in the general development, be that power of a denoted by n, or a^n, repeated as many times, as we can form different products by taking in every possible way a number n of letters from among a number m; to find the coefficient of the term containing x^{m-n} then is reduced to finding the number of these products.

138. In order to perform the problem just mentioned, it is necessary to distinguish arrangements or *permutations* from products or *combinations*. Two letters, a and b, give only one product, but admit of two arrangements, ab and ba; three letters, abc, which give only one product, admit of six arrangements (88), and so on.

To take a particular case, I will suppose the whole number of letters to be nine, namely,

$$a, b, c, d, e, f, g, h, i,$$

and that it is required to arrange them in sets of seven. It is evident, that if we take any arrangement we please, of six of these letters, $abcdef$, for example, we may join successively to it each of the three remaining letters, g, h and i; we shall then have three arrangements of seven letters, namely,

$$abcdefg, \qquad abcdefh, \qquad abcdefi.$$

What has been said of a particular arrangement of six letters, is equally true of all; we conclude therefore, that each arrangement of six letters will give three of seven, that is, as many as there remain letters, which are not employed. If therefore the number of arrangements of six letters be represented by P, we shall obtain the number consisting of seven letters by multiplying P by 3 or $9 - 6$. Representing the numbers 9 and 7 by m and n, and regarding P as expressing the number of arrangements, which can be furnished by m letters, taken $n - 1$ at a time, the same reasoning may be employed; we shall thus have for the number of arrangements of n letters,

$$P(m - (n - 1)), \qquad \text{or} \qquad P(m - n + 1).$$

This formula comprehends all the particular cases, that can occur in any question. To find, for example, the number of arrangements, that can be formed out of m letters taken two and two, or two at a time, we make $n = 2$, which gives

$$n - 1 = 1;$$

we have then

$$P = m;$$

for P will in this case be equal to the number of letters taken one at a time; there results then from this

$$m(m - 2 + 1) \quad \text{or} \quad m(m - 1),$$

for the number of arrangements taken two and two.

Again, taking

$$P = m(m - 1) \quad \text{and} \quad n = 3,$$

we find for the number of arrangements, which m letters admit of, taken three and three.

$$m(m - 1)(m - 3 + 1) = m(m - 1)(m - 2.)$$

Making

$$P = m(m - 1)(m - 2) \quad \text{and} \quad n = 4,$$

we obtain

$$m(m - 1)(m - 2)(m - 3)$$

for the number of arrangements, taken four and four. We may thus determine the number of arrangements, which may be formed from any number whatever of letters.*

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* In these arrangements it is supposed by the nature of the inquiry, that there are no repetitions of the same letter; but the theory of permutations and combinations, which is the foundation of the doctrine of chances, embraces questions in which they occur. The effect may be seen in the example we have selected, by observing, that we may write indifferently each of the 9 letters $a, b, c, d, e, f, g, h, i$, after the product of 6 letters $abcdef$. Designating therefore the number of arrangements, taken 6 at a time, by P, we shall have $P \times 9$ for the number of arrangements, taken 7 at a time. For the same reason, if P denote the number of arrangements of m letters, taken $n - 1$ at a time, that of their arrangements, when taken n at a time, will be Pm.

This being admitted, as the number of arrangements of m letters, taken one at a time, is evidently m, the number of arrangements, when taken 2 and 2, will be $m \times m$, or m^2, when taken 3 and 3, the number will be $m \times m \times m$, or m^3; and lastly, m^n will express the number of arrangements, when they are taken n and n.

139. Passing now from the number of arrangements of n letters, to that of their different products, we must find the number of arrangements, which the same product admits of. In order to this, it may be observed, that if in any of these arrangements, we put one of the letters in the first place, we may form of all the others as many permutations, as the product of $n-1$ letters admits of. Let us take, for example, the product $a\,b\,c\,d\,e\,f\,g$, composed of seven letters; we may, by putting a in the first place, write this product in as many ways, as there are arrangements in the product of six letters $b\,c\,d\,e\,f\,g$; but each letter of the proposed product may be placed first. Designating then the number of arrangements, of which a product of six letters is susceptible by Q, we shall have $Q \times 7$ for that of the arrangements of a product of seven letters. It follows from this, that if Q designate the number of arrangements, which may be formed from a product of $n-1$ letters, $Q\,n$ will express the number of arrangements of a product of n letters.

Any particular case is readily reduced to this formula; for making $n = 2$, and observing, that when there is only one letter, $Q = 1$, we have $1 \times 2 = 2$ for the number of arrangements of a product of two letters. Again, taking $Q = 1 \times 2$ and $n = 3$, we have $1 \times 2 \times 3 = 6$ for the number of arrangements of a product of three letters; further, making $Q = 1 \times 2 \times 3$, and $n = 4$, there result $1 \times 2 \times 3 \times 4$, or 24 possible arrangements in a product of four letters, and so on[C].

140. What we have now said being well understood, it will be perceived, that by dividing the whole number of arrangements obtained from m letters, taken n at a time, by the number of arrangements of which the same product is susceptible, we have for a quotient the number of the different products, which are formed by taking in all possible ways n factors among these m letters. This number will therefore be expressed by $\frac{P(m-n+1)}{Q\,n}$;* which being considered in connexion with

* It may be observed, that if we make successively

$$n = 2, \qquad n = 3, \qquad n = 4, \text{ \&c.}$$

the formula $\frac{P(m-n+1)}{Q\,n}$ becomes

$$\frac{m(m-1)}{1\,.\,2},\quad \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3},\quad \frac{m(m-1)(m-2)(m-3)}{1\,.\,2\,.\,3\,.\,4},\quad \text{\&c.}$$

what was laid down in art. 137, will give $\frac{P(m-n+1)}{Qn}a^n x^{m-n}$ for the term containing x^{m-n} in the development of $(x+a)^m$.

It is evident, that the term which precedes this, will be expressed by $\frac{P}{Q}a^{n-1}x^{m-n+1}$; for in going back towards the first term, the exponent of x is increased by unity, and that of a diminished by unity; moreover P and Q are the quantities, which belong to the number $n-1$.

141. If we make $\frac{P}{Q}=M$, the two successive terms indicated above, become

$$Ma^{n-1}x^{m-n+1} \text{ and } M\frac{(m-n+1)}{n}a^n x^{m-n}.$$

These results show how each term, in the development of $(x+a)^m$, is formed from the preceding.

Setting out from the first term, which is x^m, we arrive at the second, by making $n=1$; we have $M=1$, since x^m has only unity for its coefficient; the result then is $\frac{1\times m}{1}a\,x^{m-1}$, or $\frac{m}{1}a\,x^{m-1}$. In order to pass to the third term, we make $M=\frac{m}{1}$, and $n=2$, and we obtain $\frac{m(m-1)}{1\,.\,2}a^2x^{m-2}$. The fourth is found by supposing $M=\frac{m(m-1)}{1\,.\,2}$, and $n=3$, which gives $\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}a^3x^{m-3}$, and so on; whence we have the formula

$$(x+a)^m = x^m + \frac{m}{1}a\,x^{m-1} + \frac{m(m-1)}{1\,.\,2}a^2x^{m-2} + \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}a^3x^{m-3} + \&c.$$

which may be converted into this rule.

To pass from one term to the following, we multiply the numerical coefficient by the exponent of x *in the first, divide by the number, which marks the place of this term, increase by unity the exponent of* a, *and diminish by unity the exponent of* x.

Although we cannot determine the number of terms of this formula without assigning a particular value to m; yet, if we

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numbers, which express respectively, how many combinations may be made of any number m of things, taken two and two, three and three, four and four, &c.

observe the dependence of the terms upon each other, we can have no doubt respecting the law of their formation, to whatever extent the series may be carried. It will be seen that

$$\frac{m(m-1)(m-2)\ldots\ldots(m-n+1)}{1\ .\ 2\ .\ 3\ \ldots\ldots\ n}a^n x^{m-n}$$

expresses the term, which has n terms before it.

This last formula is called the *general term* of the series

$$x^m + \frac{m}{1}a x^{m-1} + \frac{m(m-1)}{1\ .\ 2}a^2 x^{m-2} + \&c.$$

because if we make successively

$$n=1,\quad n=2,\quad n=3,\ \&c.$$

it gives all the terms of this series.

142. Now, if $(x+a)^5$ be developed, according to the rule given in the preceding article; the first term being

$$x^5 \text{ or } a^0x^5 \quad (37),$$

the second will be

$$\frac{5}{1}a^1x^4 \quad\text{or}\quad 5ax^4,$$

the third

$$\frac{5\times 4}{2}a^2x^3 \quad\text{or}\quad 10a^2x^3,$$

the fourth

$$\frac{10\times 3}{3}a^3x^2 \quad\text{or}\quad 10a^3x^2,$$

the fifth

$$\frac{10\times 2}{4}a^4x \quad\text{or}\quad 5a^4x,$$

the sixth

$$\frac{5\times 1}{5}a^5x^0 \quad\text{or}\quad a^5.$$

Here the process terminates, because in passing to the following term it would be necessary to multiply by the exponent of x in the sixth, which is zero.

This may be shown by the formula; for the seventh term, having for a numerical coefficient,

$$\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{1\ .\ 2\ .\ 3\ .\ 4\ .\ 5\ .\ 6}$$

contains the factor $m-5$, which becomes $5-5=0$; and this same factor entering into each of the subsequent terms, reduces it to nothing.

Uniting the terms obtained above, we have

$$(x+a)^5 = x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + a^5.$$

143. Any power whatever of any binomial may be developed by the formula given in art. 141. If it were required for example to form the sixth power of $2x^3 - 5a^3$, we have only to substitute in the formula the powers of $2x^3$ and $-5a^3$ respectively for those of x and a; since, if we make

$$2x^3 = x' \text{ and } -5a^3 = a',$$

we have

$$(2x^3 - 5a^3)^6 = (x' + a')^6 =$$
$$x'^6 + 6a'x'^5 + 15a'^2x'^4 + 20a'^3x'^3$$
$$+ 15a'^4x'^2 + 6a'^5x' + a'^6 \text{ (141)},$$

and it is only necessary to substitute for x' and a' the quantities, which these letters designate. We have then

$$(2x^3)^6 + 6(-5a^3)(2x^3)^5 + 15(-5a^3)^2(2x^3)^4$$
$$+ 20(-5a^3)^3(2x^3)^3 + 15(-5a^3)^4(2x^3)^2$$
$$+ 6(-5a^3)^5(2x^3) + (-5a^3)^6,$$

or

$$64x^{18} - 960a^3x^{15} + 6000a^6x^{12}$$
$$- 20000a^9x^9 + 37500a^{12}x^6$$
$$- 37500a^{15}x^3 + 15625a^{18}.$$

The terms produced by this development are alternately positive and negative; and it is manifest, that they will always be so, when the second term of the proposed binomial has the sign $-$.

144. The formula given in art. 141, may be so expressed as to facilitate the application of it in cases analogous to the preceding

Since

$$x^{m-1} = \frac{x^m}{x}, \quad x^{m-2} = \frac{x^m}{x^2}, \quad x^{m-3} = \frac{x^m}{x^3}, \text{ \&c.}$$

the formula may be written

$$x^m + \frac{m}{1}\frac{a}{x}x^m + \frac{m(m-1)}{1 \,.\, 2}\frac{a^2}{x^2}x^m + \text{\&c.}$$

which may be reduced to

$$x^m\left\{1 + \frac{m}{1}\frac{a}{x} + \frac{m(m-1)}{1 \,.\, 2}\frac{a^2}{x^2} + \frac{m(m-1)(m-2)}{1 \,.\, 2 \,.\, 3}\frac{a^3}{x^3} + \text{\&c.}\right\},$$

by insolating the common factor x^m. In applying this formula, the several steps are, *to form the series of numbers,*

$$\frac{m}{1}, \quad \frac{m-1}{2}, \quad \frac{m-2}{3}, \quad \frac{m-3}{4}, \text{ \&c.}$$

to multiply the first by the fraction $\frac{a}{x}$, then this product by the second

and also by the fraction $\frac{a}{x}$, then again this last result by the third and by the fraction $\frac{a}{x}$, and so on; to unite all these terms, and add unity to the sum; and lastly, to multiply the whole by the factor x^m.

In the example $(2x^3 - 5a^3)^6$, we must write $(2x^3)^6$ in the place of x^m, and $-\frac{5a^3}{2x^3}$ in that of $\frac{a}{x}$. I shall leave the application of the formula as an exercise for the learner.*

145. We may easily reduce the development of the power of any polynomial whatever, to that of the powers of a binomial, as may be shown with respect to the trinomial $a + b + c$, the third power for instance being required.

First we make $b + c = m$, we then obtain,

$$(a+b+c)^3 = (a+m)^3 = a^3 + 3a^2m + 3am^2 + m^3;$$

substituting for m the binomial $b + c$, which it represents, we have

$$(a+b+c)^3 = a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3.$$

It only remains for us to develop the powers of the binomial $b + c$, and to perform the multiplications, which are indicated; we have then

$$\begin{aligned} a^3 + 3a^2b &+ 3ab^2 + b^3 \\ + 3a^2c &+ 6abc + 3b^2c \\ &+ 3ac^2 + 3bc^2 \\ &\qquad + c^3. \end{aligned}$$

Of the extraction of the roots of compound quantities.

146. Having explained the formation of the powers of compound quantities, I now pass to the extraction of their roots, beginning with the cube root of numbers.

In order to extract the cube root of numbers, we must first become acquainted with the cubes of numbers, consisting of only one figure; these are given in the second line of the following table;

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* The formula for the development of $(x + a)^m$ answers for all values of the exponent m, and is equally applicable to cases in which the exponent is fractional or negative. This property, which is very important, is demonstrated in the *Supplement* to this treatise.

1	2	3	4	5	6	7	8	9
1	8	27	64	125	216	343	512	729

and the cube of 10 being 1000, no number consisting of three figures can contain the cube of a number consisting of more than one.

The cube of a number consisting of two figures is formed in a manner analogous to that, by which we arrive at the square; for if we resolve this number into tens and units, designating the first by a, and the second by b, we have

$$(a+b)^3 = a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3.$$

Hence it is evident, that *the cube, or third power of a number composed of tens and units, contains four parts, namely; the cube of the tens, three times the square of the tens multiplied by the units, three times the tens multiplied by the square of the units, and the cube of the units.*

If it were required to find the third power of 47, by making $a = 4$ tens or 40, $b = 7$ units, we have

$$\begin{array}{lrr} & a^3 = & 64000 \\ & 3\,a^2\,b = & 33600 \\ & 3\,a\,b^2 = & 5880 \\ & b^3 = & 343 \\ \hline \text{Total} & & 103823 = 47 \times 47 \times 47. \end{array}$$

Now to go back from the cube 103823 to its root 47, we begin by observing that 64000, the cube of the 4 tens, contains no significant figure inferior to thousands; in seeking the cube of the tens therefore, we may neglect the hundreds, the tens and the units of the number 103823. Pursuing, therefore, a method similar to that employed in extracting the square root, we separate by a comma, the first three figures on the right; the greatest cube contained in 103 will be the cube of the tens. It is evident from the table, that this cube is 64, the root of which is 4; we therefore put 4 in the place assigned for the root. We then subtract 64 from 103; and by the side of the remainder 39, bring down the three last figures. The whole remainder, 39823, contains still three parts of the cube, namely, three times the square of the tens multiplied by the units, or $3\,a^2\,b$, three times the tens multiplied by the square of the units, or $3\,a\,b^2$, and the cube of the units, or b^3. If the value of the product $3\,a^2\,b$ were known, we might obtain the

$$\begin{array}{r|l} 103{,}823 & 47 \\ \hline 64 & 48 \\ \hline 39\;8{,}23 & \end{array}$$

units b, by dividing this product by $3a^2$, which is a known quantity, the tens being now found; but on the supposition, that the product $3a^2b$ is unknown, we readily perceive, that it can have no figure inferior to hundreds, since it contains the factor a^2, which represents the square of the tens; it must therefore be found in the part 398, which remains on the left of the number 39823, after the tens and units have been separated, and which contains, besides this product, the hundreds arising from the product, $3ab^2$, of the tens by the square of the units, and from the cube, b^3, of the units.

If we divide 398 by 48, which is triple the square of the tens, $3a^2$ or 3×16, we obtain 8 for the quotient; but from what precedes, it appears that we ought not to adopt this figure for the units of the root sought, until we have made trial of it by employing it in forming the three last parts of the cube, which must be contained in the remainder 39823. Making $b = 8$, we find

$$\begin{array}{lr} & 3a^2b = 38400 \\ & 3ab^2 = 7680 \\ & b^3 = 512 \\ \hline \text{Total} & 46592. \end{array}$$

As this result exceeds 39823, it is evident that the number 8 is too great for the units of the root. If we make a similar trial with 7, we find that it answers to the above conditions; 47 therefore is the root sought.

Instead of verifying the last figure of the root in the manner above described, we may raise the whole number expressed by the two figures, immediately to a cube; and this last method is generally preferred to the other. Taking the number 48 and proceeding thus, we find

$$48 \times 48 \times 48 = 110592.$$

As the result is greater than the proposed number, it is evident, that the figure 8 is too large.

147. What we have laid down in the above example may be applied to all cases, where the proposed number consists of more than three figures and less than seven. Having separated the first three figures on the right, we seek the greatest cube in the part, which remains on the left, and write its root in the usual place; we subtract this cube from the number to which it relates, and to the remainder bring down the three last figures; sepa-

rating now the tens and the units, we proceed to divide what remains on the left, by three times the square of the tens found; but before writing down the quotient as a part of the root, we verify it by raising to the cube the number consisting of the tens known, together with this figure under trial. If the result of this operation is too great, the figure for the units is to be diminished; we then proceed in the same manner with a less figure, and so on, until a root is found, the cube of which is equal to the proposed number, or is the greatest contained in this number, if it does not admit of an exact root. As we have often remainders, that are very considerable, I will here add to what has been said, a method, by which it may be soon discovered, whether or not the unit figure of the root be too small.

The cube of $a + b$, when $b = 1$, becomes that of $a + 1$,

or
$$a^3 + 3a^2 + 3a + 1,$$
a quantity, which exceeds a^3, the cube of a, by
$$3a^2 + 3a + 1.$$
Hence it follows, that *whenever the remainder, after the cube root has been extracted, is less than three times the square of the root, plus three times the root, plus unity, this root is not too small.*

148. In order to extract the root of 105823817, it may be observed, that whatever be the number of figures in this root, if we resolve it into units and tens, the cube of the tens cannot enter into the three last figures on the right, and must consequently be found in 105823. But the greatest cube contained in 105823 must have more than one figure for its root; this root then may be resolved into units and tens, and as the cube of the tens has no figure inferior to thousands, it cannot enter into the three last figures 823. If, after these are separated, there remain more than three figures on the left, we may repeat the reasoning just employed, and thus, dividing the number proposed into portions of three figures each, proceeding from right to left, and observing that the last portion may contain less than three figures, we come at length to the place occupied by the cube of the units of the highest order in the root sought.

Having thus taken the preparatory steps, we seek, by the rule given in the preceding article, the cube root of the two first por-

tions on the left, and find for the result 47; we subtract the cube of this number from the two first portions, and to the remainder 2000 bring down the following portion 817. The number 2000817 will then contain the three last parts of the cube of a number, the tens of which are 47, and the units remain to be found.

105,823,817	473
64	48
41 8,23	6627
103 823	
2 0008,17	
105 823 817	
000 000 000	

These units are therefore obtained, as in the example given in the preceding article, by separating the two last figures on the right of the remainder, and dividing the part on the left by 6627, triple the square of 47. Then making trial with the quotient 3, arising from this division, by raising 473 to a cube, we obtain for the result the proposed number, since this number is a perfect cube.

The explanation, we have given, of the above example, may take the place of a general rule. If the number proposed had contained another portion, we should have continued the operation, as we have done for the third; and it is to be recollected always, that a cipher must be placed in the root, if the number to be divided on the left of the remainder happen not to contain the number used as a divisor; we should then bring down the following portion, and proceed with it, as with the preceding.

149. *Since the cube of a fraction is found by multiplying this fraction by its square, or which amounts to the same thing, by taking the cube of the numerator and that of the denominator; reversing this process, we arrive at the root, by extracting the root of the new numerator and that of the new denominator.* The cube of $\frac{5}{6}$, for example, is $\frac{125}{216}$; taking the cube root of 125 and of 216, we find $\frac{5}{6}$.

We always proceed in this way, when the numerator and denominator are perfect cubes; but when this is not the case, we may avoid the necessity of extracting the root of the denominator, by multiplying the two terms of the proposed fraction by the square of this denominator. The denominator thence arising, will be the cube of the original denominator, and it will be only necessary then to find the root of the numerator. If we have, for example, $\frac{3}{5}$, by multiplying the two terms of this fraction by 25, the square of the denominator, we obtain

$$\frac{75}{5 \times 5 \times 5}.$$

The root of the denominator is 5; while that of 75 lies between

4 and 5. Adopting 4, we have $\frac{4}{5}$ for the cube root of $\frac{3}{5}$ to within one fifth. If a greater degree of accuracy be required, we must take the approximate root of 75, by the method I shall soon proceed to explain.

If the denominator be already a perfect square, it will only be necessary to multiply the two terms of the fraction by the square root of this denominator. Thus in order to find the cube root of $\frac{4}{9}$, we multiply the two terms by 3, the square root of 9; we thus obtain

$$\frac{12}{3 \times 3 \times 3}.$$

Taking the root of the greatest cube 8 contained in 12, we have $\frac{2}{3}$ for the root sought, within one third.

150. It follows from what has been demonstrated in art. 97, that the cube root of a number, which is not a perfect cube, cannot be expressed exactly by any fraction, however great may be the denominator; it is therefore an irrational quantity, though not of the same kind with the square root; for it is very seldom that one of them can be expressed by means of the other.

151. We may obtain the approximate cube root by means of vulgar fractions. The mode of proceeding is analogous to that given for finding the square root (103); but, as it may be readily conceived, and is besides not the most eligible, I shall not stop to explain it.

A better method of employing vulgar fractions for this purpose consists in extracting the root in fractions of a given kind. Thus, if it were required to find, for example, the cube root of 22, within a fifth part of unity, observing that the cube of $\frac{1}{5}$ is $\frac{1}{125}$, we reduce 22 to $\frac{2750}{125}$; then taking the root of 2750, so far as it can be expressed in whole numbers, we have $\frac{14}{5}$, or $2\frac{4}{5}$ for the approximate root of 22.

152. It is the practice of most persons however in extracting the cube root of a number, by approximation, to convert this number into a decimal fraction, but it is to be observed, that this fraction must be either thousandths or millionths, or of some higher denomination; because when raised to the third power, tenths become thousandths, and thousandths millionths, and in general, *the number of decimal figures found in the cube. is triple the number contained in the root.* From this it is evident, that we must place after the proposed number three times as many ciphers, as

there are decimal places required in the root. The root is then to be extracted according to the rules already given, and the requisite number of decimal figures to be distinguished in the result.

If we would find, for example, the cube root of 327, within a hundredth part of unity, we must write six ciphers after this number, and extract the root of 327000000 according to the usual method. This is done in the following manner;

327,000,000	688
216	108
1110,00	13872
3144 32	
125 680,00	
325 660 672	
1 339 328	

Separating two figures on the right of the result for decimals, we have 6,88; but 6,89 would be more exact, because the cube of this last number, although greater than 327, approaches it more nearly than that of 6,88.

If the proposed number contain decimals already, before we proceed to extract the root, we must place on the right as many ciphers, as will be necessary to render the number of decimal figures a multiple of 3. Let there be, for example, 0,07, we must write 0,070, or 70 thousandths, which gives for a root 0,4. In order to arrive at a root exact to hundredths, we must annex three additional ciphers, which gives 0,070000. The root of the greatest cube contained in 70000 being 41, that of 0,07 becomes 0,41, to within a hundredth.

153. Hitherto I have employed the formula for binomial quantities only in the extraction of the square and cube roots of numbers; this formula leads to an analogous process for obtaining the root of any degree whatever. I shall proceed to explain this process, after offering some remarks upon the extraction of roots, the exponent of which is a divisible number.

We may find the fourth root by extracting the square root twice successively; for by taking first the square root of a fourth power, a^4, for example, we obtain the square, or a^2, the square root of which is a, or the quantity sought.

It is obvious also, that the eighth root may be obtained by extracting the square root three times successively, since the square root of a^8 is a^4, and that of a^4 is a^2, and lastly, that of a^2 is a.

In the same manner it may be shown, that all roots of a degree, designated by any of the numbers 2, 4, 8, 16, 32, &c. that is, by any power of 2, are obtained by successively extracting the square root.

Roots, the exponents of which are not prime numbers, may be reduced to others of a degree less elevated; the sixth root, for example, may be found by extracting the square and afterwards the cube root. Thus, if we take a^6 and go through this process with it, we find by the first step a^3, and by the second a; we may also take first the cube root, which gives a^2, and afterwards the square root, whence we have a, as before.

154. I now proceed to treat of the general method, which I shall apply to roots of the fifth degree. The illustration will be rendered more easy, if we take a particular example; and by comparing the different steps with the methods given, for the extraction of the square and the cube root, we shall readily perceive, in what manner we are to proceed in finding roots of any degree whatever.

Let it be required then to extract the fifth root of 231554007. Now the least number, it may be observed, consisting of 2 figures, that is 10, has in its fifth power, which is 100000, six figures; we therefore conclude, that the fifth root of the number proposed contains at least two figures; this root may then be represented by $a + b$, a denoting the tens, and b the units. The expression for the proposed number will then be

$$(a+b)^5 = a^5 + 5\,a^4\,b + 10\,a^3\,b^2 + \&c.$$

I have not developed all the terms of this power, because it is sufficient, as will be seen immediately, that the composition of the two first be known.

Now it is evident, that as a^5, or the fifth power of the tens of this root, can have no figure, that falls below hundreds of thousands, it does not enter into the five last figures on the right of the proposed number; we therefore separate these five figures. If there remained more than five figures on the left, we should repeat the same reasoning, and thus separate the proposed number into portions of five figures each, proceeding from the right to the left. The last of these portions on the left, will contain the fifth power of the units of the highest order found in the root.

2315,54007	47
1024	
1291 5,4007	128

We find, by forming the fifth powers of numbers consisting of only one figure, that 2315 lies between the fifth power of 4, or 1024, and that of five, or 3125. We take therefore, 4 for th tens of the root sought; then subtracting the fifth power of thi number, or 1024, from the first portion of the proposed number we have for a remainder 1291. This remainder, together wit the following portion, which is to be brought down, must contai $5a^4b + 10a^3b^2 +$ &c. which is left, after a^5 has been subtract ed from $(a+b)^5$; but among these terms, that of the highes degree is $5a^4b$, or five times the fourth power of the tens multi plied by the units, because it has no figure, which falls below ten of thousands. In order to consider this term by itself, we sepa rate the four last figures on the right, which make no part of it, and the number 12915, remaining on the left, will contain this term, together with the tens of thousands arising from the suc ceeding terms. It is obvious therefore, that by dividing 12915 by $5a^4$, or five times the fourth power of the four tens already found, we shall only approximate the units. The fourth power of 4 is 256; five times this gives 1280; if we divide 12915 by 1280, we find 10 for the quotient, but we cannot put more tha 9 in the place of the root, and it is even necessary, before we adopt this, to try whether the whole root 49, which we thus obtain, will not give a fifth power greater than the proposed number. We find indeed by pursuing this course, that the num ber 49 must be diminished by two units, and that the actual root is 47 with a remainder 2209000; for the fifth power of 47 is 229345007; that is, the exact root of the proposed number falls between 47 and 48.

If there were another portion still, we should bring it down and annex it to the remainder, resulting from the subtraction of the fifth power found as above, from the two first portions, and pro ceed with this whole remainder, as we did with the preceding, and so on.

After what has been said, it will be easy to apply the rules, which have been given, as well in extracting the square and cube root of fractions, as in approximating the roots of imperfect powers of these degrees.

155. We may by processes, founded on the same principles, extract the roots of literal quantities. The following example

will be sufficient to illustrate the method, which is to be employed, whatever be the degree of the root required.

We found in art. 143, the sixth power of $2x^3 - 5a^3$; we shall now extract the root of this power. The process is as follows;

$$\begin{array}{l|l}
64x^{18} - 960a^3x^{15} + 6000a^6x^{12} - 20000a^9x^9 & \\
\qquad + 37500a^{12}x^6 - 37500a^{15}x^3 & 2x^3 - 5a^3 \\ \cline{2-2}
\qquad\qquad + 15625a^{18} & 192x^{15} \\
-64x^{18} & \\ \hline
\text{rem.} \quad -960a^3x^{15} + \&c. &
\end{array}$$

The quantity proposed being arranged with reference to the letter x, its first term must be the sixth power of the first term of the root arranged with reference to the same letter; taking then the sixth root of $64x^{18}$, according to the rule given in art. 129, we have $2x^3$ for the first term of the root required.

If we raise this result to the sixth power, and subtract it from the proposed quantity, the remainder must necessarily commence with the second term, produced by the development of the sixth power of the two first terms of the root. But, in the expression

$$(a+b)^6 = a^6 + 6a^5b + \&c.$$

this second term is the product of six times the fifth power of the first term of the root by the second; and if we divide it by $6a^5$, the quotient will be the second term b.

We must therefore take six times the fifth power of the first term $2x^3$ of the root, which gives

$$6 \times 32x^{15} \text{ or } 192x^{15},$$

and divide, by this quantity, the term $-960a^3x^{15}$, which is the first term of the remainder, after the preceding operation; the quotient $-5a^3$ is the second term of the root. In order to verify it, we raise the binomial $2x^3 - 5a^3$ to the sixth power, which we find is the proposed quantity itself.

If the quantity were such as to require another term in the root, we should proceed to find, after the manner above given, a second remainder, which would begin with six times the product of the fifth power of the two first terms of the root by the third, and which consequently being divided by $6(2x^3 - 5a^3)^5$, the quotient would be this third term of the root; we should then verify it by taking the sixth power of the three terms. The same course might be pursued, whatever number of terms might remain to be found.

Of equations with two terms.

156. Every equation, involving only one power of the unknown quantity, combined with known quantities, may always be reduced to two terms, one of which is made up of all those, which contain the unknown quantity, united in one expression, and the other comprehends all the known quantities collected together. This has been already shown with respect to equations of the second degree, art. 105, and may be easily proved concerning those of any degree whatever.

If we have, for example, the equation

$$a^2 x^5 - a^5 b^2 = b^4 c^3 + a c x^5,$$

by bringing all the terms involving x into one member, we obtain

$$a^2 x^5 - a c\, x^5 = b^4 c^3 + a^5 b^2$$

or

$$(a^2 - a c)\, x^5 = b^4 c^3 + a^5 b^2.$$

Now if we represent the quantities

$$a^2 - a c \text{ by } p, \quad b^4 c^3 + a^5 b^2 \text{ by } q,$$

the preceding equation becomes

$$p x^5 = q;$$

freeing x^5 from the quantity, by which it is multiplied, we have

$$x^5 = \frac{q}{p}.$$

whence we conclude

$$x = \sqrt[5]{\frac{q}{p}}.$$

In general, every equation with two terms being reduced to the form

$$p x^m = q,$$

gives

$$x^m = \frac{q}{p};$$

taking the root then of the degree m of each member, we have

$$x = \sqrt[m]{\frac{q}{p}}.$$

157. It must be observed, that if the exponent m is an odd number, the radical expression will have only one sign, which will be that of the original quantity (131).

When the exponent m is even, the radical expression will have

the double sign $\pm$; it will in this case be imaginary, if the quantity $\frac{q}{p}$ is negative, and the question will be absurd like those, of which we have seen examples in equations of the second degree (131).

See some examples.

The equation

$$x^5 = -1024$$

gives

$$x = \sqrt[5]{-1024} = -4,$$

the exponent 5 being an odd number.

The equation

$$x^4 = 625$$

gives

$$x = \pm \sqrt[4]{625} = \pm 5,$$

as the exponent 4 is even.

Lastly the equation

$$x^4 = -16,$$

which gives

$$x = \pm \sqrt[4]{-16},$$

leads only to imaginary values, because while the exponent 4 is even, the quantity under the radical sign is negative.

158. I shall here notice an analytical fact, which deserves attention on account of its utility, as well in the remaining part of the present treatise, as in the *Supplement*, and which is sufficiently remarkable in itself; it is this, that all the expressions $x - a$, $x^2 - a^2$, $x^3 - a^3$, and in general $x^m - a^m$ (m being any positive whole number), are exactly divisible by $x - a$. This is obvious, with respect to the first. We know that the second

$$x^2 - a^2 = (x + a)(x - a) \ (34),$$

and the others may be easily decomposed by division. If we divide $x^m - a^m$ by $x - a$, we obtain for a quotient

$$x^{m-1} + a\,x^{m-2} + a^2\,x^{m-3} + \&c.$$

the exponent of x, in each term, being less by unity than in the preceding, and that of a increasing in the same ratio. But instead of pursuing the operation through its several steps, I shall present immediately to the view the equation

$$\frac{x^m - a^m}{x - a} = x^{m-1} + a\,x^{m-2} + a^2\,x^{m-3} \ldots + a^{m-2}\,x + a^{m-1},$$

which may be verified by multiplying the second member by $x - a$. It then becomes

$$\begin{array}{l} x^m + a\,x^{m-1} + a^2\,x^{m-2} \ldots\ldots\ldots\ldots + a^{m-2}\,x^2 + a^{m-1}\,x \\ \quad - a\,x^{m-1} - a^2\,x^{m-2} - a^3\,x^{m-3} \ldots\ldots\ldots - a^{m-1}\,x - a^m\,; \end{array}$$

all the terms in the upper line, after the first, being the same, with the exception of the signs, as those preceding the last in the lower line, there only remains after reduction, $x^m - a^m$, that is, the dividend proposed.

It must be observed, that the term $a^2\,x^{m-2}$, in the upper line, is necessarily followed by the term $a^3\,x^{m-3}$, which is destroyed by the corresponding term in the lower line; and that, in the same manner we find, in the lower line, before the term $a^{m-1}\,x$, a term $-a^{m-2}\,x^2$, which destroys the corresponding one in the upper line. These terms are not expressed, but are supposed to be comprehended in the interval denoted by the points.

159. This leads to very important consequences, relative to the equation with two terms $x^m = \frac{q}{p}$.

If we designate by a the number, which is obtained by directly extracting the root according to the rules given in art. 154, we have

$$\frac{q}{p} = a^m \quad \text{or} \quad x^m = a^m\,;$$

transposing the second member we obtain

$$x^m - a^m = 0.$$

The quantity $x^m - a^m$ is divisible by $x - a$, and we have by the preceding article

$$x^m - a^m = (x - a)\,(x^{m-1} + a\,x^{m-2} \ldots\ldots + a^{m-2}\,x + a^{m-1}).$$

This last result, which vanishes when $x = a$, is also reduced to nothing, if we have

$$x^{m-1} + a\,x^{m-2} \ldots\ldots + a^{m-2}\,x + a^{m-1} = 0. \quad (116)\,;$$

and consequently if there exists a value of x, which satisfies this last equation, it will satisfy also the equation proposed.

These values have with unity very simple relations, which may be discovered by making $x = a\,y$; then the equation $x^m - a^m = 0$ becomes

$$a^m\,y^m - a^m = 0 \quad \text{or} \quad y^m - 1 = 0,$$

and we obtain the values of x by multiplying those of y by the number a.

The equation $y^m - 1 = 0$ gives in the first place

$$y^m = 1, \quad y = \sqrt[m]{1} = 1;$$

than by dividing $y^m - 1$ by $y - 1$, we have

$$y^{m-1} + y^{m-2} + y^{m-3} \ldots\ldots + y^2 + y + 1.$$

Taking this quotient for one of the members, and zero for the other, we form the equation on which the other values of y depend; and these values will, in the same manner, satisfy the equation

$$y^m - 1 = 0 \text{ or } y^m = 1,$$

that is, their power of the degree m will be unity.

Hence we infer the fact, singular at first view, that unity may have many *roots* besides itself. These roots, though imaginary, are still of frequent use in analysis. I can however exhibit here only those of the four first degrees, as it is only for these degrees, that we can resolve, by preceding observations, the equation

$$y^{m-1} + y^{m-2} \ldots\ldots + 1 = 0,$$

from which they are derived.

1. Let $m = 2$, we have

$$y^2 - 1 = 0,$$

whence we obtain

$$y = +1, \quad y = -1.$$

2. By making $m = 3$, we have

$$y^3 - 1 = 0,$$

whence we deduce

$$y = 1,$$

then

$$y^2 + y + 1 = 0.$$

This last equation being resolved, gives

$$y = \frac{-1 + \sqrt{-3}}{2}, \qquad y = \frac{-1 - \sqrt{-3}}{2};$$

thus we have for this degree the three roots

$$y = 1, \quad y = \frac{-1 + \sqrt{-3}}{2}, \quad y = \frac{-1 - \sqrt{-3}}{2}.$$

The two last are imaginary; but if we take the cube, forming that of the numerator, by the rule given in art. 34, and observing that the square of $\sqrt{-3}$ being -3, its cube is $-3\sqrt{-3}$, we still find $y^3 = 1$, in the same manner as when we employ the root $y = 1$.

3. Taking $m = 4$, we have

$$y^4 - 1 = 0,$$

from which we deduce

$$y = 1,$$

then $$y^3 + y^2 + y + 1 = 0.$$

We are not, at present, furnished with the means of resolving this equation; but observing that

$$y^4 - 1 = (y^2 + 1)(y^2 - 1),$$

we have successively

$$y^2 - 1 = 0, \quad y^2 + 1 = 0,$$

whence

$$y = +1, \quad y = -1, \quad y = +\sqrt{-1}, \quad y = -\sqrt{-1}.$$

Two of these values only are real; and the other two imaginary.

This multiplicity of roots of unity is agreeable to a general law of equations, according to which any unknown quantity admits of as many values, as there are units in the exponent denoting the degree of the equation, by which this unknown quantity is determined; and when the question does not admit of so many real solutions, the number is completed by purely algebraic symbols, which being subjected to the operations, that are indicated, verify the equation.

Hence it follows, that there are two kinds of expressions or values for the roots of numbers; the first, which we shall term the *arithmetical determination*, is the number which is found by the methods explained in art. 154, and which answers to each particular case; the second comprehends negative values and imaginary expressions, which we shall designate by the term *algebraic determinations*, because they consist merely in the combination of algebraic signs.

Of equations, which may be resolved in the same manner as those of the second degree.

160. These are equations, which contain only two different powers of the unknown quantity, the exponent of one of which is double that of the other. Their general formula is

$$x^{2m} + p x^m = q,$$

p and q being known quantities.

Now if we take x^m for the unknown quantity, and make $x^m = u$, we have

$$x^{2m} = u^2,$$

whence

$$u^2 + pu = q,$$

$$u = -\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2} \quad (109);$$

restoring x^m in the place of u, we have

$$x^m = -\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2},$$

an equation consisting of two terms, since the expression

$$-\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2},$$

as it implies only known operations, to be performed on given quantities, must be regarded as representing known quantities.

Designating the two values of this expression by a and a', we have

$$x^m = a \text{ and } x^m = a',$$

from which we obtain

$$x = \sqrt[m]{a} \text{ and } x = \sqrt[m]{a'}.$$

If the exponent m be even, instead of the two values given above, we shall have four, since each radical expression may take the sign $\pm$; then

$$x = +\sqrt[m]{a}, \quad x = +\sqrt[m]{a'},$$
$$x = -\sqrt[m]{a}, \quad x = -\sqrt[m]{a'},$$

and these four values will be real, if the quantities a and a' are positive.

All the values of x may be comprehended under one formula, by indicating directly the root of the two members of the equation

$$x^m = -\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2},$$

which gives

$$x = \sqrt[m]{-\tfrac{1}{2}p \pm \sqrt{q + \tfrac{1}{4}p^2}}.$$

The following question produces an equation of this kind.

161. *To resolve the number* 6 *into two such factors, that the sum of their cubes shall be* 35.

Let x be one of these factors, the other will be $\frac{6}{x}$; then taking the sum of their cubes x^3 and $\frac{216}{x^3}$, we have the equation.

$$x^3 + \frac{216}{x^3} = 35,$$

which may be reduced to

$$x^6 + 216 = 35x^3,$$

or

$$x^6 - 35x^3 = -216.$$

If we consider x^3 as the unknown quantity, we obtain, by the rule given for equations of the second degree,

$$x^3 = \tfrac{35}{2} \pm \sqrt{(\tfrac{35}{2})^2 - 216}.$$

By going through the numerical calculations, which are indicated, we find

$$(\tfrac{35}{2})^2 = \tfrac{1225}{4}$$

$$\sqrt{(\tfrac{35}{2})^2 - 216} = \sqrt{\tfrac{361}{4}} = \tfrac{19}{2},$$

and consequently

$$x^3 = \tfrac{35}{2} + \tfrac{19}{2} = \tfrac{54}{2} = 27,$$

$$x^3 = \tfrac{35}{2} - \tfrac{19}{2} = \tfrac{16}{2} = \ 8,$$

whence

$$x = \sqrt[3]{27} = 3,$$

$$x = \sqrt[3]{8} = 2.$$

The first value gives for the second factor $\frac{6}{3}$ or 2, while the second value presents $\frac{6}{2}$ or 3; we have therefore in the one case 3 and 2 for the factors sought, and in the other 2 and 3. These two solutions differ only in the order of the factors of the given number 6.

162. The equations, we have been considering, are also comprehended under the general law given in art. 159; for the values of $\sqrt[m]{a}$, $\sqrt[m]{a'}$ are to be multiplied by the roots of unity belonging to the degree denoted by the exponent m.

Applying what has been said to the equation,

$$x^6 - 35\,x^3 = -216,$$

we find the six following roots;

$$x = 1 \times 3, \qquad x = 1 \times 2,$$

$$x = \frac{-1 + \sqrt{-3}}{2} \times 3, \quad x = \frac{-1 + \sqrt{-3}}{2} \times 2,$$

$$x = \frac{-1 - \sqrt{-3}}{2} \times 3, \quad x = \frac{-1 - \sqrt{-3}}{2} \times 2,$$

of which the two first only are real.

Calculus of radical expressions.

163. The great number of cases, in which no exact root can be found, and the length of the operation necessary for obtaining it by approximation, have led algebraists to endeavour to perform, immediately upon the quantities subjected to the radical sign, the fundamental operations, intended to be performed

upon their roots. In this way we simplify the expression as much as possible, and leave the extracting of the root, which is a more complicated process, to be performed last, when the quantities are reduced to the most simple state, which the nature of the question will allow.

The addition and subtraction of dissimilar radical quantities can take place only by means of the signs + and —. For example, the sums

$$\sqrt[3]{a} + \sqrt[5]{a}, \qquad \sqrt[6]{a} + \sqrt[3]{b},$$

and the differences

$$\sqrt[3]{a} - \sqrt[5]{a}, \qquad \sqrt[3]{a} - \sqrt[3]{b},$$

can be expressed only under their present form.

The same cannot be said of the expression

$$4\,a\sqrt[3]{2b} + \sqrt[3]{16\,a^3\,b} - \frac{5c}{ad}\sqrt[3]{2\,a^6\,b},$$

because the radical quantities, of which it is composed, become similar, when they are reduced to their more simple forms, according to the method explained in art. 130. First we have

$$\sqrt[3]{16\,a^3\,b} = \sqrt[3]{8\,a^3\,.\,2b} \quad \text{or} \quad 2\,a\sqrt[3]{2b}$$

$$\sqrt[3]{2\,a^6\,b} = \sqrt[3]{a^6\,.\,2b} \quad \text{or} \quad a^2\sqrt[3]{2b};$$

the quantity therefore becomes

$$4\,a\sqrt[3]{2b} + 2\,a\sqrt[3]{2b} - \frac{5\,a^2\,c}{ad}\sqrt[3]{2b},$$

which gives, when reduced,

$$6a\sqrt[3]{2b} - \frac{5\,a\,c}{d}\sqrt[3]{2b} \quad \text{or} \quad (6\,d - 5\,c)\frac{a}{d}\sqrt[3]{2b}.$$

164. With respect to other operations the calculus of radical quantities depends upon the principle already referred to, namely; *that a product, consisting of several factors, is raised to any power by raising each of the factors to this power.* So also, by suppressing the radical sign, prefixed to a quantity, we raise this quantity to the power denoted by the exponent of this sign. For example, $\sqrt[7]{a}$ raised to the seventh power, is a simply, since this operation, being the reverse of that, which is indicated by the sign $\sqrt[7]{}$, merely restores the quantity a to its original state.

According to the principles here laid down, if, for example, in the expression

$$\sqrt[7]{a} \times \sqrt[7]{b},$$

we suppress the radical signs, the result $a\,b$ will be the s power of the above product; and taking the seventh r find

$$\sqrt[7]{a} \times \sqrt[7]{b} = \sqrt[7]{a\,b}.$$

This reasoning, which may be applied to all similar cases, that in order *to multiply two radical expressions of the same together, we must take the product of the quantities under th cal sign, observing to place it under a sign of the same degr*

We have by this rule

$$3\sqrt{2\,a\,b^3} \times 7\sqrt{5\,a^3\,b\,c} = 21\sqrt{10\,a^4\,b^4\,c} = 21\,a^2\,b^2\sqrt{10\,c};$$

$$4\sqrt{a^2 - b^2} \times \sqrt{a^2 + b^2} = 4\sqrt{(a^2 - b^2)\,(a^2 + b^2)} = 4\sqrt{a^4 - b^4};$$

$$\sqrt[5]{\frac{2\,a^9 - a^3\,b^6}{a^4 - b^4}} \times \sqrt[5]{\frac{a^2\,b^3\,c^2 + b^5\,c^2}{d^2}}$$

$$= \sqrt[5]{\frac{2\,a^9 - a^3\,b^6}{a^4 - b^4} \times \frac{a^2\,b^3\,c^2 + b^5\,c^2}{d^2}}$$

$$= \sqrt[5]{\frac{a^3\,(2\,a^6 - b^6)}{a^4 - b^4} \times \frac{b^3\,c^2}{d^2}\,(a^2 + b^2)}$$

$$= \sqrt[5]{\frac{a^3\,b^3\,c^2}{d^2} \times \frac{2\,a^6 - b^6}{a^2 - b^2}},$$

since

$$a^4 - b^4 = (a^2 + b^2)\,(a^2 - b^2).$$

165. As the seventh power of the expression $\frac{\sqrt[7]{a}}{\sqrt[7]{b}}$, for ex is $\frac{a}{b}$, it will be seen, by taking the seventh root of this la sult, that

$$\frac{\sqrt[7]{a}}{\sqrt[7]{b}} = \sqrt[7]{\frac{a}{b}}.$$

Hence *to divide a radical quantity by another of the same we must take the quotient arising from the division of the qua under the radical sign, recollecting to place it under a sign same degree.*

We find by this rule, that

$$\frac{\sqrt{6ab}}{\sqrt{3a}} = \sqrt{\frac{6ab}{3a}} = \sqrt{2b};$$

$$\frac{\sqrt{a^2 - b^2}}{\sqrt{a+b}} = \sqrt{\frac{a^2 - b^2}{a+b}} = \sqrt{a-b};$$

$$\frac{\sqrt[5]{a^4 b}}{\sqrt[5]{b^3 c^2}} = \sqrt[5]{\frac{a^4 b}{b^3 c^2}} = \sqrt[5]{\frac{a^4}{b^2 c^2}}.$$

166. It follows from the rule, given in art. 164, for the multiplication of radical quantities of the same degree, *that to raise a radical quantity to any power whatever, we have only to raise to this power the quantity under the radical sign, observing that the result must take the same sign*; thus to raise $\sqrt[5]{ab}$, for example, to the third power, is to take the product

$$\sqrt[5]{ab} \times \sqrt[5]{ab} \times \sqrt[5]{ab},$$

and as the radical signs are all of the same degree, the quantities to which they belong, are to be multiplied together, and the radical sign to be prefixed to the product, which gives

$$\sqrt[5]{5^3 b^3}.$$

In the same manner $\sqrt[7]{a^2 b^3}$ raised to the fourth power, gives $\sqrt[7]{a^8 b^{12}}$, which may be reduced to

$$ab\sqrt[7]{ab^5},$$

by resolving $a^8 b^{12}$ into $a^7 b^7 \times ab^5$, and taking the root of the factor $a^7 b^7$ (130).

It may be observed, that *when the exponent belonging to the radical sign is divisible by that of the power, to which the proposed quantity is to be raised, the operation is performed by dividing the first exponent by the second.* For example,

$$\left(\sqrt[6]{a}\right)^2 = \sqrt[3]{a},$$

because $\frac{6}{2} = 3$.

Indeed $\sqrt[6]{a}$ denotes a quantity, which is six times a factor in a, and the quantity $\sqrt[3]{a}$, which is obtained by dividing 6 by 2, being only three times a factor in a, is consequently equivalent to the product of two of the first factors, and is therefore the second power of one of these factors, or of $\sqrt[6]{a}$.

The same reasoning may be applied to all similar cases, as in the following example;

$$\left(\sqrt[12]{a^3 b}\right)^3 = \sqrt[4]{a^3 b}.$$

167. If we reverse the methods given in the preceding article, we shall be furnished with rules for extracting the roots of radical quantities.

We perceive, by attending to the rule first stated, that *if the exponents of the quantities under the radical sign are divisible by that of the root required, the operation may be performed, as if there were no radical sign, only it is to be observed, that the result must be placed under the original sign.*

We find, for example, that

$$\sqrt[3]{\sqrt[5]{a^6}} = \sqrt[5]{\sqrt[3]{a^6}} = \sqrt[5]{a^2},$$

$$\sqrt[4]{\sqrt[3]{a^4 b^8}} = \sqrt[3]{\sqrt[4]{a^4 b^8}} = \sqrt[3]{a b^2}.$$

From the second rule given in the preceding article, it is evident, that *the general method for finding the root of radical quantities, is to multiply the exponent belonging to the radical sign by that of the root, which is to be extracted.*

By this last rule, we find, that

$$\sqrt[3]{\sqrt[5]{a^4}} = \sqrt[15]{a^4},$$

In fact, $\sqrt[5]{a^4}$ is a quantity, which is five times a factor in a^4 (24, 129); but the cube root of $\sqrt[5]{a^4}$, being also three times a factor in this last quantity, is found 5×3 times or 15 times a factor in the first a^4; therefore $\sqrt[3]{\sqrt[5]{a^4}} = \sqrt[15]{a^4}$. In the same manner it might be shown, that $\sqrt[5]{\sqrt[3]{a^4}} = \sqrt[15]{a^4}$.

168. Since by multiplying the exponent of a quantity under a radical sign, by any number (166), we raise the root, which is indicated, to the power denoted by this number, and by multiplying also the exponent belonging to the radical sign, by the same number (167), we obtain for the result a root of a degree equal to that of the power, which was before formed, it is evident, that this second operation reduces the proposed quantity back to its original state.

The expression $\sqrt[5]{a^3}$, for example, may be changed into $\sqrt[35]{a^{21}}$, by multiplying the exponents 5 and 3 by 7; for multiplying the exponent of a^3 by 7, we have, making use of the radical sign, $\sqrt[5]{a^{21}}$, the seventh power of the proposed radical quantity, and multiplying by 7 the exponent 5 belonging to the radical sign in the expression $\sqrt[5]{a^{21}}$, we obtain the seventh root of the former result; this last process therefore restores the expression to its original value.

169. By this double operation, *we reduce to the same degree any number of radical quantities of different degrees, by multiplying, at the same time, the exponent belonging to each radical sign, and those of the quantities under this sign, by the product of the exponents belonging to all the other radical signs.* That the new exponents, which are thus found for the radical signs, are the same, is obvious at once, since they arise from the product of all the exponents belonging to the original radical signs; and after what has been said above, it is evident that the value of each radical quantity is the same as before.

By this rule we transform

$$\sqrt[5]{a^3 b^2} \quad \text{and} \quad \sqrt[7]{c^4 d^3},$$

into

$$\sqrt[35]{a^{21} b^{14}} \quad \text{and} \quad \sqrt[35]{c^{20} d^{15}}.$$

In the same manner the three quantities

$$\sqrt[3]{a b^2}, \quad \sqrt[5]{a^2 c^3}, \quad \sqrt[7]{b^4 c^3},$$

become respectively

$$\sqrt[105]{a^{35} b^{70}}, \quad \sqrt[105]{a^{42} c^{63}}, \quad \sqrt[105]{b^{60} c^{45}}.$$

If we meet with numbers, under the radical signs, we shall be led, in applying this rule, to raise them to the power denoted by the product of the exponents belonging to the other radical signs.

170. In the same way, we may *place under a radical sign a factor, which is without one, by raising it to the power denoted by the exponent, which accompanies this sign.*

We may change, for example

$$a^2 \text{ into } \sqrt[5]{a^{10}}, \text{ and } 2a\sqrt[3]{b} \text{ into } \sqrt[3]{8a^3 b}.$$

171. After having, by the transformation explained above, reduced any radical quantities whatever, to the same degree, we may apply to them the rules, given in articles 164 and 165, for

the multiplication and division of radical quantities of the same degree.

Let there be the general expressions

$$\sqrt[m]{a^p b^q} \times \sqrt[n]{b^r c^s};$$

we change (169)

$$\sqrt[m]{a^p b^q}, \qquad \sqrt[n]{b^r c^s},$$

into

$$\sqrt[mn]{a^{np} b^{nq}}, \qquad \sqrt[mn]{b^{mr} c^{ms}},$$

then by the rule given in art. 164, we have

$$\sqrt[mn]{a^{np}\ b^{nq}} \times \sqrt[mn]{b^{mr}\ c^{ms}} = \sqrt[mn]{a^{np}\ b^{nq+mr}\ c^{ms}},$$

for the product of the proposed radical quantities.

We have also by the rule, art. 165.

$$\frac{\sqrt[m]{a^p b^q}}{\sqrt[n]{b^r c^s}} = \frac{\sqrt[mn]{a^{np} b^{nq}}}{\sqrt[mn]{b^{mr} c^{ms}}} = \sqrt[mn]{\frac{a^{np} b^{nq}}{b^{mr} c^{ms}}} = \sqrt[mn]{\frac{a^{np} b^{nq-mr}}{c^{ms}}}.$$

Remarks on some peculiar cases, which occur in the calculus of radical quantities.

172. The rules, to which we have reduced the calculus of radical quantities, may be applied without difficulty, when the quantities employed are real. But they might lead the learner into error with regard to imaginary quantities, if they are not accompanied with some remarks upon the properties of equations with two terms.

For example, the rule laid down in art. 164, gives directly

$$\sqrt{-a} \times \sqrt{-a} = \sqrt{-a \times -a} = \sqrt{a^2};$$

and if we take $+a$ for $\sqrt{a^2}$, we evidently come to an erroneous result, for the product $\sqrt{-a} \times \sqrt{-a}$, being the square of $\sqrt{-a}$, must be obtained by suppressing the radical sign, and is therefore equal to $-a$.

Bézout has obviated this difficulty, by observing, that when we do not know by what method the square a^2 has been formed, we must assign for its root both $+a$ and $-a$; but when, by means of steps already taken, we know which of these two quantities multiplied by itself produced a^2, we are not allowed, in

going back to the root, to take the other quantity. This is evidently the case with respect to the expression $\sqrt{-a} \times \sqrt{-a}$; here we know, that the quantity a^2, contained under the radical sign in the expression $\sqrt{a^2}$, arises from $-a$ multiplied by $-a$; the ambiguity therefore is prevented, and it will be readily seen, that in taking the root, we are limited to $-a$.

The difficulty above mentioned would present itself in regard to the product $\sqrt{a} \times \sqrt{a}$, if we were not led, by the circumstance, of there being no negative sign in the expression, to take immediately the positive value of $\sqrt{a^2}$. In this case, since a^2 arises from $+a$ multiplied by $+a$, its root must necessarily be $+a$.

There can be no doubt with respect to examples of the kind we have been considering; but there are cases, which can be clearly explained only by attending to the properties of equations with two terms.

173. If, for example, it were required to find the product $\sqrt[4]{a}\sqrt{-1}$; reducing the second of these radical expressions to the same degree with the first (169), we have

$$\sqrt[4]{a} \times \sqrt[4]{(-1)^2} = \sqrt[4]{a} \times \sqrt[4]{+1} = \sqrt[4]{a},$$

a result, which is real, although it appears evident, that the quantity $\sqrt[4]{a}$ multiplied by the imaginary quantity $\sqrt{-1}$, ought to give an imaginary product. It must not be supposed however, that the expression $\sqrt[4]{a}$ is in all respects false, but only that it is to be taken in a very peculiar sense.

In fact, $\sqrt[4]{a}$, considered algebraically, being the expression for the unknown quantity x, in the equation with two terms,

$$x^4 - a = 0,$$

admits of four different values (159); for if we make $a = \alpha^4$, by taking α to represent the numerical value of $\sqrt[4]{a}$, considered independently of its sign, or the arithmetical determination of this quantity, we have the four values

$$\alpha \times +1, \quad \alpha \times -1, \quad \alpha \times +\sqrt{-1}, \quad \alpha \times -\sqrt{-1},$$

the third of which is precisely the product proposed.

By a little attention, it will be readily perceived, whence the ambiguity, of which we have been speaking arises. The second power $+1$ of the quantity -1 under the radical sign, as it may

arise, as well from $+1 \times +1$, as from -1×-1, causes the quantity $\sqrt[4]{1}$ to have two values, which are not found in $\sqrt{-1}$.

In general, the process by which the product $\sqrt[m]{a} \times \sqrt[n]{b}$ is formed, is reduced to that of raising this product to the power mn; for if we represent it by z, that is, if we make

$$\sqrt[m]{a} \times \sqrt[n]{b} = z,$$

by raising the two members of this equation, first to the power m, we have

$$a\sqrt[n]{b^m} = z^m,$$

again, raising it to the power n, we obtain

$$a^n b^m = z^{mn}.$$

This product therefore, being determined only by means of its power of the degree mn, or by an equation of this degree with two terms, must have mn values (159). This will be perceived at once, if we reflect that the expressions $\sqrt[m]{a}$ and $\sqrt[n]{b}$, being nothing but the values of the unknown quantities x and y, in the equations with two terms

$$x^m - a = 0, \quad y^n - b = 0,$$

and consequently admitting of m and of n determinations, we have, by uniting the several m determinations of x, with the several n determinations of y, mn determinations of the product required.

When we are employed upon real quantities, there is no difficulty in finding the values, because the number of those, that are real, is never more than two (157), which differ only in the sign.

174. If we use the transformation explained in art. 159, the difficulty will be confined to the roots of $+1$ and -1; for if we make $x = \alpha t$ and $y = \beta u$, α and β denoting the numerical values of $\sqrt[m]{a}$, $\sqrt[n]{b}$ considered without regard to the sign, the equations

$$x^m \mp a = 0, \qquad y^n \mp b = 0,$$

become

$$t^m \mp 1 = 0, \qquad u^n \mp 1 = 0,$$

whence

$$xy = \sqrt[m]{\pm a} \times \sqrt[n]{\pm b} = \alpha\beta tu = \alpha\beta \sqrt[m]{\pm 1} \times \sqrt[n]{\pm 1};$$

in which $\alpha\beta$ represents the product of the numbers $\sqrt[m]{a}$, $\sqrt[n]{b}$, or the

arithmetical determination of the root of the degree $m\,n$ of the number $a^n\,b^m$.

If we would give a determinate value to the product of the radical quantities $\sqrt[m]{\pm a}$, $\sqrt[n]{\pm b}$, by fixing the degree of the radical signs, we must obtain from the equations

$$t^m \mp 1 = 0, \quad u^n \mp 1 = 0,$$

the several expressions for $\sqrt[m]{\pm 1}$, $\sqrt[n]{\pm 1}$, and combine them in a suitable manner.

To conclude, these operations are not often required, except in some very simple cases, of which the following are the principal;

1. $$\sqrt{-a} \times \sqrt{-b} = \sqrt{a} \times \sqrt{b}\,(\sqrt{-1} \times \sqrt{-1});$$

I suppress the radical sign in the expression $\sqrt{-1}$, and obtain

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{ab} \times -1 = -\sqrt{ab}.$$

2. $$\sqrt[4]{-a} \times \sqrt[4]{-b} = \sqrt[4]{ab}\,(\sqrt[4]{-1})^2;$$

I do not here multiply -1 by -1, because this would lead to the ambiguity mentioned in art. 173; but observing, that the square of the fourth root is simply the square root, we have

$$\sqrt[4]{-a} \times \sqrt[4]{-b} = \sqrt[4]{ab} \times \sqrt{-1}.$$

3. $$\sqrt[6]{-a} \times \sqrt[6]{-b} = \sqrt[6]{ab} \times (\sqrt[6]{-1})^2 = \sqrt[6]{ab} \times \sqrt[3]{-1}$$
$$= \sqrt[6]{ab} \times -1 = -\sqrt[6]{ab}.$$

The results will be thus found to be alternately real and imaginary.

Calculus of fractional exponents.

175. If we substitute, in the place of the radical signs, their corresponding fractional exponents (132), and apply immediately the rules for the exponents, we shall obtain the same results, as those furnished by the methods employed in the calculus of radical quantities.

If we transform, for example,

$$\sqrt[5]{a^3\,b^2}, \quad \sqrt[5]{a^3\,c^2}$$

into

$$a^{\frac{3}{5}}\,b^{\frac{2}{5}}, \quad a^{\frac{3}{5}}\,c^{\frac{2}{5}},$$

we have

$$\sqrt[5]{a^3 b^2} \times \sqrt[5]{a^3 c^2} = a^{\frac{3}{5}} b^{\frac{2}{5}} \times a^{\frac{3}{5}} c^{\frac{2}{5}} =$$
$$a^{\frac{3}{5}+\frac{3}{5}} b^{\frac{2}{5}} c^{\frac{2}{5}} = a^{\frac{6}{5}} b^{\frac{2}{5}} c^{\frac{2}{5}};$$

then since $\frac{6}{5} = 1 + \frac{1}{5}$, and consequently

$$a^{\frac{6}{5}} = a^{1+\frac{1}{5}} = a \times a^{\frac{1}{5}} \quad (25),$$

and $a^{\frac{1}{5}} b^{\frac{2}{5}} c^{\frac{2}{5}}$ is equivalent to $\sqrt[5]{a b^2 c^2}$, we have

$$\sqrt[5]{a^3 b^2} \times \sqrt[5]{a^3 c^2} = a \sqrt[5]{a b^2 c^2},$$

a result which is not only exact, but is reduced to its most s ple form.

Let there be the general example $\sqrt[m]{a^p b^q} \times \sqrt[n]{b^r c^s}$; the rad expressions here employed may be transformed into

$$a^{\frac{p}{m}} b^{\frac{q}{m}}, \quad b^{\frac{r}{n}} c^{\frac{s}{n}},$$

we then have, according to the rules for the exponents, (25),

$$a^{\frac{p}{m}} b^{\frac{q}{m}} \times b^{\frac{r}{n}} c^{\frac{s}{n}} = a^{\frac{p}{m}} b^{\frac{q}{m}+\frac{r}{n}} c^{\frac{s}{n}}.$$

Now in order to add the fractions $\frac{q}{m}, \frac{r}{n}$, we must reduce t to the same denominator; and to give uniformity to the res we must do the same with respect to the fractions $\frac{p}{m}, \frac{s}{n}$; we tain, by this means,

$$a^{\frac{np}{mn}} b^{\frac{nq+mr}{mn}} c^{\frac{ms}{mn}};$$

and placing this result under the radical sign, we have

$$\sqrt[m]{a^p b^q} \times \sqrt[n]{b^r c^s} = \sqrt[nm]{a^{np} b^{nq+mr} c^{ms}}.$$

176. The manner of performing division is equally simple, have for example

$$\frac{\sqrt[5]{a^3 b^2}}{\sqrt[5]{a^4 c}} = \frac{a^{\frac{3}{5}} b^{\frac{2}{5}}}{a^{\frac{4}{5}} c^{\frac{1}{5}}} = \frac{b^{\frac{2}{5}}}{a^{\frac{4}{5}-\frac{3}{5}} c^{\frac{1}{5}}} \quad (38),$$

which may be reduced to

$$\frac{b^{\frac{2}{5}}}{a^{\frac{1}{5}} c^{\frac{1}{5}}}:$$

this placed under the radical sign becomes

$$\frac{\sqrt[5]{a^3 b^2}}{\sqrt[5]{a^4 c}} = \sqrt[5]{\frac{b^2}{a c}}.$$

We have in general,

$$\frac{\sqrt[m]{a^p b^q}}{\sqrt[n]{b^r c^s}} = \frac{a^{\frac{p}{m}} b^{\frac{q}{m}}}{b^{\frac{r}{n}} c^{\frac{s}{n}}} = \frac{a^{\frac{p}{m}} b^{\frac{q}{m} - \frac{r}{n}}}{c^{\frac{s}{n}}};$$

reducing the fractional exponents to the same denominator, in order to perform the subtraction, which is required, we find

$$\frac{\sqrt[m]{a^p b^q}}{\sqrt[n]{b^r c^s}} = \frac{a^{\frac{np}{mn}} b^{\frac{nq-mr}{mn}}}{c^{\frac{ms}{mn}}} = \sqrt[mn]{\frac{a^{np} b^{nq-mr}}{c^{ms}}}.$$

It is obvious, that the reduction of fractional exponents to the same denominator, answers here to the reduction of radical expressions to the same degree, and leads to precisely the same results (171).

177. It is also very evident, by the rule given in art. 127, that

$$\left(\sqrt[m]{a^p}\right)^n = \left(a^{\frac{p}{m}}\right)^n = a^{\frac{np}{m}} = \sqrt[m]{a^{np}},$$

and by the rule laid down in art. 129, that

$$\sqrt[n]{\sqrt[m]{a^p}} = \sqrt[n]{a^{\frac{p}{m}}} = a^{\frac{p}{mn}} = \sqrt[mn]{a^p}.$$

The calculus of fractional exponents affords one of the most remarkable examples of the utility of signs, when well chosen. The analogy, which prevails among exponents both fractional and entire, renders the rules, that are to be followed with respect to the latter, applicable also to the former; but a particular investigation is necessary in each case, when we use the sign $\sqrt{}$, because it has no connexion with the operation that is indicated. The further we advance in algebra the more fully shall we be convinced of the numerous advantages, which arise from the notation by exponents, introduced by Descartes.

General theory of equations.

178. Equations of the first and second degree are, properly speaking, the only ones, which admit of a complete solution; but there are general properties of equations of whatever degree, by which we are able to solve them, when they are numerical, and

which lead to many conclusions of use in the higher parts of algebra. These properties relate to the particular form, which every equation is capable of assuming.

An equation in its most general form must contain all the powers of the unknown quantity, from that of the degree of the equation to the first degree, multiplied each by some known quantity, together with one term wholly known.

A general equation of the fifth degree, for example, contains all the powers of the unknown quantity, from the first to the fifth; and if there are several terms involving the same power of the unknown quantity, we must suppose them to be united in one, according to the method given for equations of the second degree, art. 108. All the terms of the equation are then to be brought into one member, as in the article above referred to; the other member will necessarily be zero; and when the first term is negative, it is rendered positive by changing the signs of all the terms of the equation.

In this way we obtain an expression similar to the following;

$$n x^5 + p x^4 + q x^3 + r x^2 + s x + t = 0,$$

in which it is to be observed, that the letters n, p, q, r, s, t, may represent negative as well as positive numbers; then dividing the whole by n, in order that the first term may have only unity for its coefficient, and making

$$\frac{p}{n} = P, \quad \frac{q}{n} = Q, \quad \frac{r}{n} = R, \quad \frac{s}{n} = S, \quad \frac{t}{n} = T,$$

we have

$$x^5 + P x^4 + Q x^3 + R x^2 + S x + T = 0.$$

In future I shall suppose, that equations have always been prepared as above, and shall represent the general equation of any degree whatever by

$$x^n + P x^{n-1} + Q x^{n-2} \ldots\ldots + T x + U = 0.$$

The interval denoted by the points may be filled up, when the exponent n takes a determinate value.

Every quantity or expression, whether real or imaginary, which, put in the place of the unknown quantity x in an equation prepared as above, renders the first member equal to zero, and which consequently satisfies the question, is called the *root of the proposed equation*; but as the inquiry does not at present relate to powers, this acceptation of the term *root* is more general, than that, in which it has hitherto been used (90, 129).

179. **Take a proposition analogous to those given in articles 116 and 159, and one which may be regarded as fundamental.**

If the root of any equation whatever

$$x^n + P x^{n-1} + Q x^{n-2} \ldots\ldots + T x + U = 0,$$

be represented by a, *the first member of this equation may be exactly divided by* x — a.

Indeed, since a is one value of x, we have necessarily

$$a^n + P a^{n-1} + Q a^{n-2} \ldots\ldots + T a + U = 0,$$

and consequently

$$U = -a^n - P a^{n-1} - Q a^{n-2} \ldots\ldots - T a,$$

so that the equation proposed is precisely the same as

$$\left.\begin{array}{l} x^n + P x^{n-1} + Q x^{n-2} \ldots\ldots + T x \\ -a^n - P a^{n-1} - Q a^{n-2} \ldots\ldots - T a \end{array}\right\} = 0,$$

which may be reduced to

$$\left.\begin{array}{l} x^n - a^n + P(x^{n-1} - a^{n-1}) + Q(x^{n-2} - a^{n-2}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + T(x - a) \end{array}\right\} = 0.$$

As the quantities

$$x^n - a^n,\quad x^{n-1} - a^{n-1},\quad x^{n-2} - a^{n-2}, \ldots\ldots x - a,$$

are each divisible by $x - a$ (158), it is evident, that the first member of the proposed equation is made up of terms, all of which are divisible by this quantity, and may consequently be divided by $x - a$, as the enunciation of the proposition requires.*

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* D'Alembert has proved the same proposition in the following manner.

If we conceive the first member of the proposed equation to be divided by $x - a$, and the operation continued until all the terms involving x are exhausted, the remainder, if there be any, cannot contain x. If we represent this remainder by R, and the quotient to which we arrive by Q, we have necessarily

$$x^n + P x^{n-1} \ldots\ldots + \&c. = Q(x - a) + R.$$

Now if we substitute a in the place of x, the first member is reduced to nothing, since a is the value of x; the term $Q(x - a)$ is also nothing, because the factor $x - a$ becomes zero; we must therefore have $R = 0$, and it is so, independently of the substitution of a; for as this remainder does not contain x, the substitution cannot take place, and it still preserves the value it had before.

Hence it follows, that in every case, $R = 0$, and that consequently

$$x^n + P x^{n-1} + Q x^{n-2}, \&c.$$

is exactly divisible by $x - a$.

180. To form the quotient we have only to substitute for the quantities

$$x^n - a^n,\ \ x^{n-1} - a^{n-1},\ \ x^{n-2} - a^{n-2}, \ldots\ldots x - a,$$

the quotients, which are obtained by dividing these quantities by $x - a$, and which are respectively

$$\begin{array}{r} x^{n-1} + a\,x^{n-2} + a^2\,x^{n-3} \ldots\ldots + a^{n-1}, \\ x^{n-2} + \ a\,x^{n-3} \ldots\ldots + a^{n-2}, \\ x^{n-3} \ldots\ldots + a^{n-3}, \\ \ldots\ldots\ldots\ldots\ldots \\ + 1. \end{array}$$

Arranging the result with reference to the powers of x, we have

$$\begin{array}{r} x^{n-1} + a\,x^{n-2} + a^2\,x^{n-3} \ldots\ldots + \ a^{n-1} \\ + P\,x^{n-} + P\,a\,x^{n-3} \ldots\ldots + P\,a^{n-2} \\ + \ Q\,x^{n-3} \ldots\ldots + Q\,a^{n-3} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + T. \end{array}$$

181. It is evident from the rules of division simply, that if the first member of the equation

$$x^n + P\,x^{n-1} + Q\,x^{n-2} + \&c. = 0,$$

be divided by $x - a$, the quotient obtained will be exhibited under the following form

$$x^{n-1} + P'\,x^{n-2} + Q'\,x^{n-3} + \&c.$$

P', Q', &c. representing known quantities different from P, Q, &c. we have then

$$x^n + P\,x^{n-1} + \&c. = (x - a)\ (x^{n-1} + P'\,x^{n-2} + \&c.)\ ;$$

and according to what was observed in art. 116, the proposed equation may be verified in two ways, namely, by making

$$x - a = 0 \quad \text{or} \quad x^{n-1} + P'\,x^{n-2} + \&c. = 0.$$

Now if the equation

$$x^{n-1} + P'\,x^{n-2} + \&c. = 0$$

has a root b, its first member will be divisible by $x - b$; we have then

$$x^{n-1} + P'\,x^{n-2} + \&c. = (x - b)\ (x^{n-2} + P''\,x^{n-3} + \&c.),$$

and consequently

$$x^n + P\,x^{n-1} + \&c. = (x - a)\ (x - b)\ (x^{n-2} + P''\,x^{n-3} + \&c.);$$

the equation proposed may therefore be verified in three ways, namely, by making

$$x - a = 0, \text{ or } x - b = 0, \text{ or } x^{n-2} + P''\,x^{n-3} + \&c. = 0.$$

If the last of these equations has a root c, its first member may still be decomposed into two factors

$$x - c, \quad x^{n-3} + P''' x^{n-4} + \&c. = 0;$$

we then have

$$x^n + P x^{n-1} + \&c.$$
$$= (x - a)(x - b)(x - c)(x^{n-3} + P''' x^{n-4} + \&c.);$$

from which it is obvious, that the proposed equation may be verified in four ways, namely, by making

$$x - a = 0, \quad x - b = 0, \quad x - c = 0, \quad x^{n-3} + P''' x^{n-4} + \&c. = 0.$$

Pursuing the same reasoning, we obtain successively factors of the degrees

$$n - 4, \quad n - 5, \quad n - 6, \ \&c.;$$

and if each of these factors being put equal to zero, is susceptible of a root, the first member of the proposed equation is reduced to the form

$$(x - a)(x - b)(x - c)(x - d) \ldots\ldots (x - l),$$

that is, it is decomposed into as many factors of the first degree, as there are units in the exponent n, which denotes the degree of the equation.

The equation

$$x^n + P x^{n-1} + \&c. = 0,$$

may be verified in n ways, namely, by making

$$x - a = 0, \quad \text{or} \quad x - b = 0, \quad \text{or} \quad x - c = 0, \quad \text{or} \quad x - d = 0,$$

or lastly, $\qquad x - l = 0.$

It is necessary to observe, that these equations are to be regarded as true only when taken one after the other, and there arise manifest contradictions from the supposition, that they are true at the same time. In fact, from the equation $x - a = 0$, we obtain $x = a$, while $x - b = 0$ gives $x = b$, results, which are inconsistent, when a and b are unequal quantities.

182. If the first member of the proposed equation,

$$x^n + P x^{n-1} + \&c. = 0,$$

be decomposed into n factors of the first degree

$$x - a, \ x - b, \ x - c, \ x - d, \ldots\ldots x - l,$$

it cannot be divided by any other expression of this degree. Indeed, if it were possible to divide it by a binomial $x - \alpha$, different from the former ones, we should have

$$x^n + P x^{n-1} + \&c. = (x - \alpha)(x^{n-1} + p\, x^{n-2} + \&c.)$$

and consequently

$$(x-a)(x-b)(x-c)(x-d)\ldots\ldots(x-l)$$
$$=(x-\alpha)(x^{n-1}+p\,x^{n-2}+\&c.)\,;$$

now by changing x into α, this becomes

$$(\alpha-a)(\alpha-b)(\alpha-c)(\alpha-d)\ldots\ldots(\alpha-l)$$
$$=(\alpha-\alpha)(\alpha^{n-1}+p\,\alpha^{n-2}+\&c.)$$

The second member vanishes by means of the factor $\alpha-\alpha$, which is nothing; this is not the case with respect to the first, which is the product of factors, all of which are different from zero, so long as α differs from the several roots $a, b, c, d \ldots l$. The supposition we have made then is not true; therefore *an equation of any degree whatever does not admit of more binomial divisors of the first degree, than there are units in the exponent denoting its degree, and consequently cannot have a greater number of roots.**

183. An equation regarded as the product of a number of factors

$$x-a,\quad x-b,\quad x-c,\quad x-d,\ \&c.$$

equal to the exponent of its degree, may take the form of the product exhibited in art. 135, with this modification, that the terms will be alternately positive and negative.

If we take four factors, for example, we have

$$\begin{array}{llll}
x^4 - a\,x^3 & +a\,b\,x^2 & -a\,b\,c\,x & +a\,b\,c\,d=0. \\
\quad\ - b\,x^3 & +a\,c\,x^2 & -a\,b\,d\,x & \\
\quad\ - c\,x^3 & +a\,d\,x^2 & -a\,c\,d\,x & \\
\quad\ - d\,x^3 & +b\,c\,x^2 & -b\,c\,d\,x & \\
 & +b\,d\,x^2 & & \\
 & +c\,d\,x^2 & &
\end{array}$$

The second terms of the binomials $x-a$, $x-b$, $x-c$, &c. being the roots of the equation, taken with the contrary sign, the properties enumerated in art. 135, and proved generally in art. 136, will, in the present case, be as follows,

The coefficient of the second term, taken with the contrary sign, will be the sum of the roots;

The coefficient of the third term will be the sum of the products of the roots, taken two and two;

The coefficient of the fourth term, taken with the contrary sign, will be the sum of the products of the roots, multiplied three and

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* This demonstration is taken from the *Annales de Mathématiques*, published by M. Gergonne. See vol. iv. pp. 209, 210, note.

three, and so on, the signs of the coefficients of the even terms being changed;

The last term, subject also to this law, *will be the product of all the roots.*

Making, for example, the product of the three factors

$$x - 5,\ x + 4,\ x + 3,$$

equal to zero, we form the equation

$$x^3 + 2x^2 - 22x - 60 = 0,$$

the roots of which are

$$+5,\ -4,\ -3;$$

we have for their sum

$$5 - 4 - 3 = -2;$$

for the sum of their products, taken two and two,

$$+5 \times -4 + 5 \times -3 - 4 \times -3 = -20 - 15 + 12 = -23,$$

and for the product of the three roots

$$+5 \times -4 \times -3 = 60.$$

In this way we form the coefficients 2, -23, -60, changing the signs of those for the second and fourth terms.

If we make the product of the factors

$$x - 2,\quad x - 3 \text{ and } x + 5,$$

equal to zero, the equation thence arising

$$x^3 - 19x + 30 = 0,$$

as it has no term involving x^2, the power immediately inferior to that of the first term, *wants the second term;* and the reason is, that the sum of the roots, which, taken with the contrary sign, forms the coefficient of this term, is here

$$2 + 3 - 5,$$

or zero, or in other words, the sum of the positive roots is equal to that of the negative.*

184. We have proved (182), that an equation, considered as arising from the product of several simple factors, or factors of the first degree, can contain only as many of these factors, as there are units in the exponent n denoting the degree of this equation; but if we combine these factors two and two, we form quantities of the second degree, which will also be factors of the proposed equation, the number of which will be expressed by

$$\frac{n(n-1)}{1\,.\,2}\quad (140).$$

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* See note at the end of this treatise.

For example, the first member of the equation

$$\begin{array}{lllll} x^4 & -a x^3 & +a b x^2 & -a b c x & +a b c d = 0 \\ & -b x^3 & +a c x^2 & -a b d x & \\ & -c x^3 & +a d x^2 & -a c d x & \\ & -d x^3 & +b c x^2 & -b c d x & \\ & & +b d x^2 & & \\ & & +c d x^2 & & \end{array}$$

being the product of

$$(x-a) \times (x-b) \times (x-c) \times (x-d),$$

may be decomposed into factors of the second degree, in the six following ways;

$$\begin{array}{l} (x-a)\ (x-b) \times (x-c)\ (x-d) \\ (x-a)\ (x-c) \times (x-b)\ (x-d) \\ (x-a)\ (x-d) \times (x-b)\ (x-c) \\ (x-b)\ (x-c) \times (x-a)\ (x-d) \\ (x-b)\ (x-d) \times (x-a)\ (x-c) \\ (x-c)\ (x-d) \times (x-a)\ (x-b); \end{array}$$

whence it appears, that an equation of the fourth degree may have six divisors of the second.

By combining the simple factors three and three, we form quantities of the third degree for divisors of the proposed equation; for an equation of the degree n the number will be

$$\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3},$$

and so on.

Of elimination among equations exceeding the first degree.

185. The rule given in art. 78, or the method pointed out in art. 84, is sufficient, in all cases, for eliminating in two equations an unknown quantity, which does not exceed the first degree, whatever may be the degree of the others; and the rule of art. 78 is applicable, even when the unknown quantity is of the first degree in only one of the proposed equations.

If we have, for example, the equations

$$a x^2 + b x y + c y^2 = m^2$$
$$x^2 + x y = n^2,$$

taking, in the second, the value of y, which will be

$$y = \frac{n^2 - x^2}{x},$$

and substituting this value and its square, in the place of y and y^2 in the first equation, we obtain a result involving only x.

186. If both of the proposed equations involved the second power of each of the two unknown quantities, the above method could be applied in resolving only one of the equations, either with respect to x or y.

Let there be, for example, the equations

$$a x^2 + b x y + c y^2 = m^2,$$
$$x^2 + y^2 = n^2 ;$$

the second gives

$$y = \pm \sqrt{n^2 - x^2} ;$$

Substituting this value of y and its square in the first, we obtain

$$a x^2 \pm b x \sqrt{n^2 - x^2} + c(n^2 - x^2) = m^2.$$

Our purpose appears to be answered, since we have arrived at a result, which does not involve the unknown quantity y, but we are unable to resolve the equation containing x, without reducing it to a rational form, by making the radical sign, under which the unknown quantity is found, to disappear.

It will be readily seen, that if this radical expression stood alone in one member, we might make the radical sign to disappear by raising this member to a square. Collecting together all the rational terms then in one member, by transposing the terms $\pm b x \sqrt{n^2 - x^2}$ and m^2, we have

$$a x^2 + c(n^2 - x^2) - m^2 = \mp b x \sqrt{n^2 - x^2} ;$$

taking the square of each member, we form the equation

$$\left.\begin{array}{l} a^2 x^4 + c^2 (n^2 - x^2)^2 + m^4 \\ + 2acx^2(n^2 - x^2) - 2am^2 x^2 - 2cm^2(n^2 - x^2) \end{array}\right\} = b^2 x^2 (n^2 - x^2),$$

which contains no radical expression.

The method, we have just employed for making the radical sign to disappear, deserves attention, on account of the frequent occasion we have to apply it; it consists in *insolating the quantity found under the radical sign, and then raising the two members of the proposed equation to the power denoted by the degree of this sign.*

187. The complicated nature of this process, which increases in proportion to the number of radical expressions, added to the difficulty of resolving one of the proposed equations with reference to one of the unknown quantities, a difficulty, which is often insurmountable in the present state of algebra, has led those, who have cultivated this science, to seek a method of effecting the

elimination without this; so that the resolution of the equation shall be the last of the operations required for the solution of the problem.

In order to render the operation more simple, we reduce equations with two unknown quantities to the form of equations with only one, by presenting only that, which we wish to eliminate. If we have, for example,

$$x^2 + axy + bx = cy^2 + dy + e,$$

we bring all the terms into one member, and arrange them with reference to x; the equation then becomes

$$x^2 + (ay + b)x - cy^2 - dy - e = 0;$$

abridging this, by making

$$ay + b = P, \quad -cy^2 - dy - e = Q,$$

we have

$$x^2 + Px + Q = 0.$$

The general equation of the degree m with two unknown quantities must contain all the powers of x and y, which do not exceed this degree, as well as those products, in which the sum of the exponents of x and y does not exceed m; this equation then may be represented thus;

$$x^m + (a+by)x^{m-1} + (c+dy+ey^2)x^{m-2} + (f+gy+hy^2+ky^3)x^{m-3}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+(p+qy+ry^2 \ldots + uy^{m-1})x + p' + q'y + r'y^2 \ldots + v'y^m = 0.$$

No coefficient is assigned to x^m in this equation, because we may always, by division, free any term of an equation we please, from the number, by which it is multiplied. Now if we make

$$a + by = P, \quad c + dy + ey^2 = Q, \quad f + gy + hy^2 + ky^3 = R,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$p + qy \ldots\ldots + uy^{m-1} = T, \quad p' + q'y \ldots\ldots + v'y^m = U,$$

the above equation takes the following form

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} \ldots\ldots + Tx + U = 0.$$

188. It should be observed, that we may immediately eliminate x in the two equations of the second degree,

$$x^2 + Px + Q = 0, \quad x^2 + P'x + Q' = 0,$$

by subtracting the second from the first. This operation gives

$$(P - P')x + Q - Q' = 0,$$

whence

$$x = -\frac{Q - Q'}{P - P'};$$

substituting this value in one of the two proposed equations, the first for example, we find

$$\frac{(Q-Q')^2}{(P-P')^2}-\frac{P(Q-Q')}{P-P'}+Q=0;$$

naking the denominators to disappear, we have

$$(Q-Q')^2-P(P-P')(Q-Q')+Q(P-P')^2=0,$$

hen developing the two last terms, and making the reduction

$$(Q-Q')^2+(P-P')(PQ'-QP')=0.$$

We have then only to substitute for P, Q, P' and Q', the particuar values, which answer to the case under consideration.

189. Before proceeding further I shall show, how we may etermine, whether the value of any one of the unknown quanties satisfies, at the same time, the two equations proposed. In rder to make this more clear, I shall take a particular example; the reasoning employed will however be of a general nature.

Let there be the equations

$$x^3+3x^2y+3xy^2-98=0\ldots\ldots(1)$$

$$x^2+4xy-2y^2-10=0\ldots\ldots(2)$$

which we shall suppose furnished by a question, that gives $y=3$. In order to verify this supposition, we must substitute 3 in the ace of y, in the proposed equation; we have then

$$x^3+9x^2+27x-98=0\ldots\ldots(a)$$

$$x^2+12x-28=0\ldots\ldots(b)$$

uations, which must present the same value of x, if that, which as been assigned to y, be correct. If the value of x be represented by α, the equation (a) and the equation (b) will, according to what has been proved in art. 179, both of them be divisible y $x-\alpha$; they must therefore have a common divisor, of which $-\alpha$ forms a part; and in fact, we find for this common divior $x-2$ (48); we have therefore $\alpha=2$. Thus the value $y=3$ ulfils the conditions of the question, and corresponds to $x=2$.

If there remained any doubt, whether or not the common divior of the equations (a) and (b) must give the value of x, we might remove it by observing, that these equations reduce themelves to

$$(x^2+11x+49)(x-2)=0,$$

$$(x+14)(x-2)=0,$$

om which it is evident, that they are verified by putting 2 in e place of x.

190. The method I have just explained, for finding the value f x, when that of y is known, may be employed immediately in e elimination of x.

Indeed, if we take the equations (1) and (2), and go through the process necessary for determining, whether they have a common divisor involving x, instead of finding one, we arrive at a remainder, which contains only the unknown quantity y and numbers, that are given; and it is evident, that if we put in the place of y its value 3, this remainder will vanish, since by the same substitution, the equations (1) and (2), become the equations (a) and (b). which have a common divisor. Forming an equation therefore, by taking this remainder and zero for the two members, we express the condition, which the values of y must fulfil, in order that the two given equations may admit, at the same time, of the same value for x.

The adjoining table presents the several steps of the operation relative to the equations,

$$x^3 + 3\,x^2\,y + 3\,x\,y^2 - 98 = 0$$

$$x^2 + 4\,x\,y - 2\,y^2 - 10 = 0,$$

on which we have been employed in the preceding article. We find for the last divisor,

$$(9\,y^2 + 10)\,x - 2\,y^3 - 10\,y - 98\,;$$

and the remainder, being taken equal to zero, gives

$$43\,y^6 + 345\,y^4 - 1960\,y^3 + 750\,y^2 - 2940\,y - 4302 = 0,$$

an equation, which admits, besides the value $y = 3$ given above, of all the other values of y, of which the question proposed is susceptible.

The remainder above mentioned being destroyed, that preceding the last becomes the common divisor of the equations proposed; and being put into an equation, gives the value of x when that of y is introduced. Knowing, for example, that $y = 3$, we substitute this value in the quantity

$$(9\,y^2 + 10)\,x - 2\,y^3 - 10\,y - 98\,;$$

then taking the result for one member, and zero for the other, we have the equation of the first degree

$$91\,x - 182 = 0, \quad \text{or} \quad x = 2.$$

191. The operation, to which the above equations have been subjected, furnishes occasion for several important remarks. First, it may happen that the value of y reduces the remainder preceding the last to nothing; in this case, the next higher remainder, or that, which involves the second power of x, becomes the common divisor of the two proposed equations. Introducing then into this the value of y, and putting it equal to zero,

$$\begin{array}{l|l}
x^3+3x^2y+3y^2x-98 & x^2+4xy-2y^2-10 \\ \hline
-x^3-4x^2y+2y^2x+10x & x-y \\ \hline
\quad -x^2y+5y^2x+10x\ -98 & \\
\quad +x^2y+4y^2x-\ 2y^3-10y & \\ \hline
\text{1st. rem.}\ldots\ldots+(9y^2+10)x-\ 2y^3-10y-98 &
\end{array}$$

$$\begin{array}{rl|l}
 & x^2+\ 4xy\ -\ 2y^2-\ 10 & (9y^2+10)x-2y^3-10y-98 \\ \hline
\text{or rather} & (9y^2+10)x^2+36xy^3-18y^4-110y^2-100 & (9y^2+10)x-2y^3-10y-98 \\
 & \qquad\qquad +40xy & \\ \hline
 & -(9y^2+10)x^2+\ 2xy^3\ +98x & x+38y^3+50y+98 \\
 & \qquad\qquad +10xy & \\ \hline
 & \qquad\qquad +38xy^3-18y^4-110y^2-100 & \\
 & \qquad\qquad +50xy & \\
 & \qquad\qquad +98x &
\end{array}$$

$$\begin{array}{rl}
\text{or rather} & (38y^3+50y+98)(9y^2+10)x-162y^6-1170y^4-2000y^2-1000 \\
 & -(38y^3+50y+98)(9y^2+10)x+\ 76y^6+\ 480y^4+3920y^3+\ 500y^2+5880y+9604 \\ \hline
\text{2d. rem.} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ -\ 86y^6-\ 690y^4+3920y^3-1500y^2+5880y+8604
\end{array}$$

Putting this remainder equal to zero, then dividing all its terms by 2, and changing the signs in order to make the first term positive, we have

$$43y^6+345y^4-1960y^3+750y^2-2940y-4302=0.$$

we have an equation of the second degree, involving only x, the two values of which will correspond to the known value of y. If this value still reduce to nothing the remainder of the second degree, we must go back to the preceding, or that into which th third power of x enters, because this, in the case under consideration, becomes the common divisor of the two proposed equations; and the value of y will correspond to the three values of x. In general, we must go back until we arrive at a remainder, which is not destroyed by substituting the value of y.

It may sometimes happen, that there is no remainder, or that the remainder contains only known quantities.

In the first case, the two equations have a common divisor independently of any determination of y; they assume then the following form,

$$P \times D = 0, \quad Q \times D = 0,$$

D being the common divisor. It is evident, that we satisfy both the equations at the same time, by making, in the first place $D = 0$; and this equation will enable us to determine one of the unknown quantities by means of the other, when the factor D contains both; but if it contains only given quantities and x, this unknown quantity will be determinate, and the other will remain wholly indeterminate. With respect to the factors, which do not contain x, they are found by what is laid down in art. 50.

Next, if we make at the same time

$$P = 0, \quad Q = 0,$$

we have still two equations, which will furnish solutions of the question proposed.

Let there be, for example,

$$(a\,x + b\,y - c)\,(m\,x + n\,y - d) = 0,$$
$$(a'\,x + b'\,y - c')\,(m\,x + n\,y - d) = 0;$$

by supposing, first, the second factor, common to the two equations, to be nothing, we have with respect to the unknown quantities x and y only the equation

$$m\,x + n\,y - d = 0,$$

and in this view the question will be indeterminate; but if we suppress this factor, we are furnished with the equations

$$a\,x + b\,y - c = 0, \qquad a'\,x + b'\,y - c' = 0,$$

or

$$a\,x + b\,y = c, \qquad a'\,x + b'\,y = c';$$

and in this case the question will be determinate, since we have as many equations as unknown quantities.

When the remainder contains only given quantities, the two proposed equations are contradictory; for the common divisor, by which it is shown, that they may both be true at the same time, cannot exist except by a condition, which can never be fulfilled.[(D)] This case corresponds to that mentioned in art. 68, relative to equations of the first degree.*

192. If then we have any two equations

$$x^m + P x^{m-1} + Q x^{m-2} + R x^{m-3} \ldots\ldots + T x + U = 0,$$
$$x^n + P' x^{n-1} + Q' x^{n-2} + R' x^{n-3} \ldots\ldots + V' x + Z' = 0,$$

where the second unknown quantity y is involved in the coefficients P, Q, &c. P', Q', &c. in seeking the greatest common divisor of their first members, we resolve them into other more simple expressions, or come to a remainder independent of x, which must be made equal to zero.

This remainder will form the *final equation* of the question proposed, if it does not contain factors foreign to this question; but it very often begins with polynomials involving y, by which the highest power of x in the several quantities, that have been successively employed as divisors, is multiplied, and we arrive at a result more complicated than that, which is sought, should be. In order to avoid being led into error with respect to the values of y arising from these factors, the idea, which first presents itself, is to substitute immediately in the equations proposed each of the values furnished by the equation involving y only; for all the values, which give a common divisor to these equations, necessarily belong to the question, and the others must be excluded. It will be perceived also, that the final equation will

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* It will be readily perceived, by what precedes, that the problem for obtaining the final equation from two equations with two unknown quantities, is, in general, determinate; but the same final equation answers to an infinite variety of systems of equations with two unknown quantities. Reversing the process, by which the greatest common divisor of two quantities is obtained, we may form these systems at pleasure; but as this inquiry relates to what would be of little use in the elementary parts of mathematics, and would lead me into tedious details, I shall not pursue it here. Researches of this nature must be left to the sagacity of the intelligent reader, who will not fail, as occasion offers, of arriving at a satisfactory result.

become incomplete, if we suppress in the operation any factor involving y; but all these circumstances together occasion some inconvenience in the application of the above method,* and lead me to prefer the method given by Euler, which I shall explain in the following article.

193. Let there be the equations

$$x^3 + Px^2 + Qx + R = 0,$$
$$x^4 + P'x^3 + Q'x^2 + R'x + S' = 0;$$

representing by $x - \alpha$ the factor, which must be common to both, when y is determinate in a proper sense, we may consider the first as the product of $x - \alpha$ by the factor of the second degree, $x^2 + px + q$, and the second as the product of $x - \alpha$ by the factor of the third degree $x^3 + p'x^2 + q'x + r'$, p and q, p', q' and r' being indeterminate coefficients. We have then

$$x^3 + Px^2 + Qx + R = (x - \alpha)(x^2 + px + q),$$
$$x^4 + P'x^3 + Q'x^2 + R'x + S' = (x - \alpha)(x^3 + p'x^2 + q'x + r').$$

Exterminating the binomial $(x - \alpha)$, in the same manner as an unknown quantity of the first degree (84), we find

$$(x^3 + Px^2 + Qx + R)(x^3 + p'x^2 + q'x + r') =$$
$$(x^4 + P'x^3 + Q'x^2 + R'x + S')(x^2 + px + q);$$

a result, which must verify itself without any particular value being assigned to x; this cannot take place however, unless the first member be composed of the same terms as the second; we must therefore, after performing the multiplications, which are indicated, put the coefficients belonging to each power of x in one member respectively equal to those belonging to the same power in the other. In this way we obtain the following equations;

$$P + p' = P' + p \qquad Rp' + Qq' + Pr' = S' + R'p + Q'q$$
$$Q + Pp' + q' = Q' + P'p + q \qquad Rq' + Qr' = S'p + R'q$$
$$R + Qp' + Pq' + r' = R' + Q'p + P'q \qquad Rr' = S'q.$$

As we have here six equations, and only five indeterminate quantities, namely p, q, p', q' and r', all of which are of the first

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* On this subject see a memoir of M. Bret, in the 15th number, of *Journal de l'Ecole Polytechnique*, also one of M. Lefébure, 3d number, vol. ii. of the *Correspondance* of the same school.

degree, these quantities may be exterminated; we shall thus arrive at an equation, which, involving only the quantities P, Q, R, P', Q', R', and S', will express a condition necessarily implyed in the conditions of the question, and which consequently will be the final equation in y.*

Should this equation be identical, it follows, that the proposed equations have at least one factor of the form $x-\alpha$, whatever y may be; on the contrary, if the final equation contain only known quantities, the proposed equations are contradictory.

When the final equation takes place, we obtain the factor $x-\alpha$ by dividing the first of the proposed equations by the polynomial x^2+px+q; we find for the quotient

$$x+P-p,$$

and neglect the remainder, because it must necessarily be reduced to nothing, when we substitute in the place of y a value obtained from the final equation. Putting the above quotient equal to zero, we find

$$x=p-P,$$

* The method of Euler, explained here, amounts to multiplying each of the proposed equations by a factor, the coefficients of which are indeterminate, putting the products equal, and disposing the coefficients in such a manner, that the terms containing the unknown quantity destroy each other. In this form it is presented in his *Introduction to the analysis of infinites.* The exponent, which denotes the degree of the products, being designated by k, that of the factors is $k-m$ for the equation of the degree m, and $k-n$ for that of the degree n. The first term of each of these factors, having unity for a coefficient, the one contains $k-m$ indeterminate coefficients, and the other $k-n$. The sum of the products contains a number k of terms involving x; but it is necessary to destroy $k-1$ terms only, because that, which contains the highest power of x, vanishes of itself. It follows from this, that the whole number $2k-m-n$ of indeterminate coefficients must be equal to $k-1$, and consequently $k=m+n-1$; we must therefore multiply the equation of the degree m by a factor of the degree $n-1$, that of the degree n by a factor of the degree $m-1$, and put the products equal, term to term, a method similar to that given in the text. It may be observed, that this former method of Euler contains the germ of that developed by Bézout in his *Théorie des Equations Algébriques.*

and this value of x will be known, or at least will be expressed by means of y, if we substitute for p its value deduced from the equations of the first degree, formed above.

This expression assumes, in general, a fractional from, so that we have $x=\frac{M}{N}$, or $Nx-M=0$; and it may be seen in this case, that the values of y, which would cause M and N to vanish at the same time, would verify the preceding equation independently of x; this takes place in consequence of the fact, that by means of these values, the proposed equations would acquire a common factor of a degree above the first. It would not be difficult to go back to the immediate conditions in which this circumstance is implied; but the limits I have prescribed to myself in the present treatise do not permit me to enter into details of this kind.

194. Now let there be the equations

$$x^2+Px+Q=0, \quad x^2+P'x+Q'=0;$$

the factors, by which $x-\alpha$ is multiplied, will be here of the first degree, or $x+p$ and $x+p'$ simply; in this case

$$R=0,\ R'=0,\ S'=0,\ q=0,\ q'=0,\ r'=0,$$

and we have

$$\left.\begin{array}{l} P+p' \quad =P'+p \\ Q+Pp'=Q'+P'p \\ \quad Qp'=Q'p \end{array}\right\}\text{or}\left\{\begin{array}{l} \quad p-p' \quad =P-P' \\ P'p-Pp'=Q-Q' \\ Q'p-Qp'=0. \end{array}\right.$$

From the two first equations we obtain

$$p=\frac{(P-P')P-(Q-Q')}{P-P'};$$

$$p'=\frac{(P-P')P'-(Q-Q')}{P-P'}.$$

Substituting these values in the third, we have

$$(P-P)'Q'P-(Q-Q')Q'=(P-P')P'Q-(Q-Q')Q$$

or $$(P-P')(PQ'-QP')+(Q-Q')^2=0.$$

Now if in the equation

$$x=p-P;$$

we put, in the place of p, its value found above, we have

$$x=-\frac{Q-Q'}{P-P'}.$$

195. In order to aid the learner, I shall indicate the operations necessary for eliminating x in the two equations

$$x^3+Px^2+Qx+R=0, \quad x^3+P'x^2+Q'x+R'=0.$$

n this case, we have

$$S' = 0, \quad r' = 0 \quad (193),$$

nd are furnished with these five equations;

$$\begin{aligned} P + p' &= P' + p, \\ Q + P p' + q' &= Q' + P' p + q, \\ R + Q p' + P q' &= R' + Q' p + P' q, \\ R p' + Q q' &= R' p + Q' q \\ R q' &= R' q, \end{aligned}$$

hich may take the following form

$$\begin{aligned} p - p' &= P - P', \\ P' p - P p' + q - q' &= Q - Q', \\ Q' p - Q p' + P' q - P q' &= R - R', \\ R' p - R p' + Q' q - Q q' &= 0, \\ R' q - R q' &= 0, \end{aligned}$$

We may, by the rules given in art. 88, obtain immediately om any four of these equations, the values of the unknown iantities p, p', q and q'; but the simple form, under which the rst and the last of the equations are presented, enables us to rrive at the result, by a more expeditious method. In order to bridge the expressions, we make

$$P - P' = e, \quad Q - Q' = e', \quad R - R' = e'';$$

nd proceed to deduce from the first and last of the proposed quations,

$$p' = p - e, \quad q' = \frac{R' q}{R};$$

ien substituting these values in the three others, and making ie denominator R to disappear, we have

$$\begin{aligned} (P' - P) R p + (R - R') q &= R (e' - P e) \ldots (\mathrm{a}), \\ (Q' - Q) R p + (R P' - P R') q &= R (e'' - Q e) \ldots (\mathrm{b}), \\ (R' - R) R p + (R Q' - Q R') q &= - R^2 e \ldots\ldots\ldots (\mathrm{c}). \end{aligned}$$

now we obtain, from the equations (a) and (b), the values of p nd q (88), and suppress the factor R, which will be common to ie numerators and the denominator, we have

$$p = \frac{(e' - P e)(R P' - P R') - (R - R')(e'' - Q e)}{(P' - P)(R P' - P R') - (R - R')(Q' - Q)},$$

$$q = \frac{(P' - P)(e'' - Q e) R - R (e' - P e)(Q' - Q)}{(P' - P)(R P' - P R') - (R - R')(Q' - Q)};$$

utting these values in the equation (c), we obtain a final equaion, divisible by R, and which may be reduced to

$$(R'-R)[(e'-Pe)(RP'-PR')-(R-R')(e''-Qe)]$$
$$+(RQ'-QR')[(P'-P)(e''-Qe)-(e'-Pe)(Q'-Q)]$$
$$=-Re[(P'-P)(RP'-PR')-(R-R')(Q'-Q)],$$

it only remains then to substitute for the letters e, e' e'' the quantities they represent.

196. If we have the three unknown quantities x, y and z, and are furnished with an equal number of equations distinguished by (1), (2) and (3); in order to determine these unknown quantities, we may combine, for example, the equation (1) with (2) and with (3), to eliminate x, and then extermine y from the two results, which are obtained. But it must be observed, that by this *successive* elimination, the three proposed equations do not concur, in the same manner, to form the final equation; the equation (1) is employed twice, while (2) and (3) are employed only once; hence the result, to which we arrive, contains a factor foreign to the question (84.) Bézout, in his *Théorie des Equations*, has made use of a method, which is not subject to this inconvenience, and by which he proves, that *the degree of the final equation, resulting from the elimination among any number whatever of complete equations, containing an equal number of unknown quantities, and quantities of any degrees whatever; is equal to the product of the exponents, which denote the degree of these equations.* M. Poisson, has given a demonstration of the same proposition more direct and shorter than that of Bézout; but the preliminary information, which it requires, will not permit me to explain it here; it will be found in the *Supplement*. At present I shall observe simply, that it is easy to verify this proposition in the case of the final equations presented in articles 194 and 195. If we suppose the proposed equations given in those articles to be complete, the unknown quantity y enters of the first degree into P and P', of the second degree into Q and Q', of the third into R and R'; hence it follows, that e will be of the first degree, e' of the second, and e'' of the third, and that the terms of the highest degree found in the products indicated in the final equation given in art. 194, will have 4, or 2 . 2, for an exponent, and those of the final equation art. 195 will have 9 or 3 . 3.

Of commensurable roots, and the equal roots of numerical equations.

197. HAVING made known the most important properties of algebraic equations, and explained the method of eliminating the unknown quantities, when several occur, I shall proceed to the numerical resolution of equations with only one unknown quantity, that is, to the finding of their roots, when their coefficients are expressed by numbers.*

I shall begin by showing, that *when the proposed equation has only whole numbers for its coefficients, and that of its first term is unity, its real roots cannot be expressed by fractions, and consequently can be only whole numbers, or numbers, that are incommensurable.*

In order to prove this, let there be the equation

$$x^n + Px^{n-1} + Qx^{n-2} \ldots\ldots + Tx + U = 0,$$

in which we substitute for x an irreducible fraction $\frac{a}{b}$; the equation then becomes

$$\frac{a^n}{b^n} + P\frac{a^{n-1}}{b^{n-1}} + Q\frac{a^{n-2}}{b^{n-2}} \ldots\ldots + T\frac{a}{b} + U = 0;$$

reducing all the terms to the same denominator, we have

$$a^n + Pa^{n-1}b + Qa^{n-2}b^2 \ldots\ldots + Tab^{n-1} + Ub^n = 0,$$

which is equivalent to

$$a^n + b\left(Pa^{n-1} + Qa^{n-2}b \ldots\ldots + Tab^{n-2} + Ub^{n-1}\right) = 0.$$

The first member of this last equation consists of two entire parts, one of which is divisible by b, and the other is not (98), since it is supposed, that the fraction $\frac{a}{b}$ is reduced to its most simple form, or that a and b have no common divisor; one of these parts cannot therefore destroy the other.

198. After what has been said, we shall perceive the utility of making the fractions of an equation to disappear, or of rendering its coefficients entire numbers, in such a manner however,

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* There is no general solution for degrees higher than the fourth; properly speaking, it is only that for the second degree, which can be regarded as complete. The expressions for the roots of equations of the third and fourth degree are very complicated, subject to exceptions, and less convenient in practice than those, which I am about to give; I shall resume the subject in the *Supplement*.

that the first term may have only unity for its coefficient. This is done *by making the unknown quantity proposed equal to a new unknown quantity divided by the product of all the denominators of the equation*, then reducing all the terms to the same denominator, by the method given in art. 52.

Let there be, for example, the equation

$$x^3+\frac{a\,x^2}{m}+\frac{b\,x}{n}+\frac{c}{p}=0;$$

we take $x=\frac{y}{m\,n\,p}$, and introducing this expression for x into the proposed equation, we obtain

$$\frac{y^3}{m^3\,n^3\,p^3}+\frac{a\,y^2}{m^3\,n^2\,p^2}+\frac{b\,y}{m\,n^2\,p}+\frac{c}{p}=0;$$

as the divisor of the first term contains all the factors found in the other divisors, we may multiply by this divisor and thus reduce each term to its most simple expression; we find then

$$y^3+a\,n\,p\,y^2+b\,m^2\,n\,p^2\,y+c\,m^3\,n^3\,p^2=0.$$

If the denominators m, n, p, have common divisors, it is only necessary to divide y by the least number, which can be divided at the same time by all the denominators. As these methods of simplifying expressions will be readily perceived, I shall not stop to explain them; I shall observe only, that if all the denominators were equal to m, it would be sufficient to make $x=\frac{y}{m}$.

The proposed equation, which would be in this case,

$$x^3+\frac{a\,x^2}{m}+\frac{b\,x}{m}+\frac{c}{m}=0,$$

then becomes

$$\frac{y^3}{m^3}+\frac{a\,y^2}{m^3}+\frac{b\,y}{m^2}+\frac{c}{m}=0,$$

and we have

$$y^3+a\,y^2+b\,m\,y+m^2\,c=0.$$

It is evident, that the above operation amounts to multiplying all the roots of the proposed equations by the number m, since $x=\frac{y}{m}$ gives $y=m\,x$.

199. Now since, if a be the root of the equation

$$x^n+P\,x^{n-1}+Q\,x^{n-2}\ldots\ldots+T\,x+U=0,$$

we have

$$U=-a^n-P\,a^{n-1}-Q\,a^{n-2}\ldots\ldots-T\,a \quad (179),$$

ws, that a is necessarily one of the divisors of the entire r U, and consequently, when this number has but few divi- re have only to substitute them successively in the place of he proposed equation, in order to determine, whether or not uation has any root among whole numbers.

'e have, for example, the equation

$$x^3 - 6x^2 + 27x - 38 = 0,$$

numbers

$$1, \quad 2, \quad 19, \quad 38,$$

e only divisors of the number 38, we make trial of these, n their positive and negative state; and we find, that the number $+2$ only satisfies the proposed equation, or that . We then divide the proposed equation by $x-2$; put- he quotient equal to zero, we form the equation

$$x^2 - 4x + 19 = 0,$$

ots of which are imaginary; and resolving this we find, ie proposed equation has three roots,

$$x = 2, \quad x = 2 + \sqrt{-15}, \; x = 2 - \sqrt{-15}.$$

. The method just explained, for finding the entire number, satisfies an equation, becomes impracticable, when the last f this equation has a great number of divisors; but the on,

$$U = -a^n - Pa^{n-1} - Qa^{n-2} \ldots - Ta,$$

hes new conditions, by means of which the operation may y much abridged. In order to make the process more I shall take, as an example, the equation

$$x^4 + Px^3 + Qx^2 + Rx + S = 0.$$

ɔot being constantly represented by a, we have,

$$a^4 + Pa^3 + Qa^2 + Ra + S = 0,$$
$$S = -Ra - Qa^2 - Pa^3 - a^4,$$

which we obtain

$$\frac{S}{a} = -R - Qa - Pa^2 - a^3.$$

evident from this last equation, that $\frac{S}{a}$ must be a whole er.

nging R into the first member, we have

$$\frac{S}{a} + R = -Qa - Pa^2 - a^3;$$

abridging the expression by making $\frac{S}{a}+R=R'$, and dividing the two members of the equation

$$R'=-Qa-Pa^2-a^3$$

by a, we have

$$\frac{R'}{a}=-Q-Pa-a^2,$$

whence we conclude, that $\frac{R'}{a}$ must also be a whole number.

Transposing Q, and making $\frac{R'}{a}+Q=Q'$, then dividing the two members by a, we obtain

$$\frac{Q'}{a}=-P-a,$$

whence we infer, that $\frac{Q'}{a}$ must be a whole number.

Lastly, bringing P into the first member, making $\frac{Q'}{a}+P=P'$, and dividing by a, we have

$$\frac{P'}{a}=-1.$$

Putting together the above mentioned conditions, we shall perceive, that the number a will be the root of the proposed equation, if it satisfy the equations

$$\frac{S}{a}+R=R',$$
$$\frac{R'}{a}+Q=Q',$$
$$\frac{Q'}{a}+P=P',$$
$$\frac{P'}{a}+1=0,$$

in such a manner, as to make R', Q' and P' whole numbers.

Hence it follows, that in order to determine, whether one of the divisors a of the last term S can be a root of the proposed equation, we must,

1st. *Divide the last term by the divisor* a, *and add to the quotient the coefficient of the term involving* x ;

2d. *Divide this sum by the divisor* a, *and add to the quotient the coefficient of the term involving* x^2 ;

3d. *Divide this sum by the divisor* a, *and add to the quotient the coefficient of the term involving* x^3 ;

4th. *Divide this sum by the divisor* a, *and add to the quotient unity, or the coefficient of the term involving* x^4; *the result will become equal to zero, if* a *is, in fact, the root.*

The rules given above are applicable, whatever be the degree of the equation; it must be observed however, that the result will not become equal to zero, until we arrive at the first term of the proposed equation.*

201. In applying these rules to a numerical example, we may conduct the operation in such a manner as to introduce the several trials with all the divisors of the last term, at the same time.

For the equation

$$x^4 - 9x^3 + 23x^2 - 20x + 15 = 0,$$

the operation is, as follows;

$+15,$	$+5,$	$+3,$	$+1,$	$-1,$	$-3,$	$-5,$	$-15,$
$+1,$	$+3,$	$+5,$	$+15,$	$-15,$	$-5,$	$-3,$	$-1,$
$-19,$	$-17,$	$-15,$	$-5,$	$-35,$	$-25,$	$-23,$	$-21,$
		$-5,$	$-5,$	$+35,$			
		$+18,$	$+18,$	$+58,$			
		$+6,$	$+18,$	$-58,$			
		$-3,$	$+9,$	$-67,$			
		$-1,$	$+9,$	$+67,$			
		$0.$					

All the divisors of the last term 15 are arranged, in the order of magnitude, both with the sign $+$ and $-$, and placed in the same line; this is the line occupied by the divisors a.

The second line contains the quotients arising from the number 15 divided successively by all its divisors; this is the line for the quantities $\frac{S}{a}$.

The third line is formed by adding to the numbers found in the

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* It would not be difficult to prove by means of the formula for the quotients given in art. 180, that the quantities $\frac{S}{a}, \frac{R'}{a}, \frac{Q'}{a}$, taken with the contrary sign, and with the order inverted, are the coefficients of the quotient arising from the polynomial

$$x^4 + Px^3 + Qx^2 + Rx + S$$

divided by $x - a$, and which is consequently

$$x^3 - \frac{Q'}{a}x^2 - \frac{R'}{a}x - \frac{S}{a}.$$

preceding the coefficient -20, by which x is multiplied; this is the line for the quantities $R' = \frac{S}{a} + R$.

The fourth line contains the quotients of the several numbers in the preceding divided by the corresponding divisors; this is the line for the quantities $\frac{R'}{a}$. In forming this line, we neglect all the numbers, which are not entire.

The fifth line results from the numbers, written in the preceding, added to the number 23, by which x^2 is multiplied; this line contains the quantities Q'.

The sixth line contains the quotients arising from the numbers in the preceding divided by the corresponding divisors; it comprehends the quantities $\frac{Q'}{a}$.

The seventh line comprehends the several sums of the numbers in the preceding, added to the coefficient -9, by which x^3 is multiplied; in this line are found the quantities $\frac{Q'}{a} + P$.

Lastly, the eighth line is formed by dividing the several numbers in the preceding by the corresponding divisors; it is the line for $\frac{P}{a}$. As we find -1 only in the column, at the head of which $+3$ stands, we conclude, that the proposed equation has only one commensurable root, namely $+3$; it is therefore divisible by $x - 3$.*

The divisors $+1$ and -1 may be omitted in the table, as it is easier to make trial of them, by substituting them immediately in the proposed equation.

202. Again, let there be, for example,

$$x^3 - 7x^2 + 36 = 0.$$

Having ascertained, that the numbers $+1$ and -1 do not satisfy this equation, we form the table subjoined, according to the preceding rules, observing that, as the term involving x is wanting in this equation, x must be regarded as having 0 for a coefficient; we must therefore suppress the third line, and deduce the fourth immediately from the second.

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* Forming the quotient according to the preceding note, we find

$$x^3 - 6x^2 + 5x - 5.$$

$+36, +18, +12, +9, +6, +4, +3, +2, -2, -3, -4, -6, -9, -12, -18, -36$
$+1, +2, +3, +4, +6, +9, +12, +18, -18, -12, -9, -6, -4, -3, -2, -1$
....

$+1, \quad +4, +9, +9, +4, \quad +1$
$-6, \quad -3, +2, +2, -3, \quad -6$
$-1, \quad -1, \qquad -1, +1, \quad +1$
$0, \quad 0, \qquad 0.$

We find in this example three numbers, which fulfil all the conditions, namely, $+6+3$ and -2. Thus we obtain, at the same time, the three roots, which the proposed equation admits of; we conclude then, that it is the product of three simple factors, $x-6$, $x-3$ and $x+2$.

203. It may be observed, that there are literal equations, which may be transformed, at once, into numerical ones.

If we have, for example,

$$y^3 + 2py^2 - 33p^2y + 14p^3 = 0,$$

making $y = px$, we obtain

$$p^3x^3 + 2p^3x^2 - 33p^3x + 14p^3 = 0,$$

a result, which is divisible by p^3, and may be reduced to

$$x^3 + 2x^2 - 33x + 14 = 0.$$

As the commensurable divisor of this last equation is $x+7$, which gives $x = -7$, we have

$$y = -7p.$$

The equation involving y is among those, which are called *homogeneous equations*, because taken independently of the numerical coefficients, the several terms contain the same number of factors.*

204. When we have determined one of the roots of an equation, we may take for an unknown quantity the difference between this root and any one of the others; by this means we arrive at an equation of a degree inferior to that of the equation proposed, and which presents several remarkable properties.

Let there be the general equation

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} \ldots\ldots + Tx + U = 0,$$

and let $a, b, c, d,$ &c. be its roots; substituting $a+y$ in the place of x, and developing the powers, we have

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* For a more full account of the *commensurable divisors* of equations, the reader is referred to the third part of the *Élémens d'Algèbre* of Clairaut. This geometer has treated of literal as well as numerical equations.

$$\left.\begin{array}{llll}
a^m & +ma^{m-1}y & +\dfrac{m(m-1)}{2} & a^{m-2}y^2+\ldots+y^m \\
+Pa^{m-1} & +(m-1)Pa^{m-2}y & +\dfrac{(m-1)(m-2)}{2} & Pa^{m-3}y^2+\ldots \\
+Qa^{m-2} & +(m-2)Qa^{m-3}y & +\dfrac{(m-2)(m-3)}{2} & Qa^{m-4}y^2+\ldots \\
+Ra^{m-3} & +(m-3)Ra^{m-4}y & +\dfrac{(m-3)(m-4)}{2} & Ra^{m-5}y^2+\ldots \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
+Ta & +Ty & & \\
+U & & &
\end{array}\right\}=0.$$

The first column of this result, being similar to the proposed equation, vanishes of itself, since a is one of the roots of this equation; we may therefore suppress this column, and divide all the remaining terms by y; the equation then becomes

$$\left.\begin{array}{lll}
m\,a^{m-1}+ & \dfrac{m(m-1)}{2} & a^{m-2}y+\ldots+y^{m-1} \\
+(m-1)\,P\,a^{m-2}+ & \dfrac{(m-1)(m-2)}{2} & P\,a^{m-3}y+\ldots \\
+(m-2)\,Q\,a^{m-3}+ & \dfrac{(m-2)(m-3)}{2} & Q\,a^{m-4}y+\ldots \\
+(m-3)\,R\,a^{m-4}+ & \dfrac{(m-3)(m-4)}{2} & R\,a^{m-5}y+\ldots \\
\ldots\ldots & \ldots\ldots & \ldots\ldots \\
+T & &
\end{array}\right\}=0.$$

This equation has evidently for its $m-1$ roots

$$y=b-a,\quad y=c-a,\quad y=d-a\ldots\ldots\ \&c.$$

I shall represent it by

$$\mathcal{A}+\frac{B}{2}y+\frac{C}{2.3}y^2\ldots\ldots\ldots+y^{m-1}=0\ldots\ldots\ldots(d),$$

abridging the expressions, by making

$$m\,a^{m-1}+(m-1)\,P\,a^{m-2}+(m-2)\,Q\,a^{m-3}\ldots\ldots+T=\mathcal{A}$$
$$m\,(m-1)\,a^{m-2}+(m-1)\,(m-2)\,P\,a^{m-3}\ldots\ldots\quad=B$$

&c.

and I shall designate by V the expression

$$a^m+P\,a^{m-1}+Q\,a^{m-2}\ldots\ldots\ldots+T\,a+U.$$

205. If the proposed equation has two equal roots; if we have, for example, $a=b$, one of the values of y, namely, $b-a$, becomes nothing; the equation (d) will therefore be verified, by supposing $y=0$; but upon this supposition all the terms vanish, except the known term $\mathcal{A}$; this last must therefore be nothing of itself; the value of a must therefore satisfy, at the same time, the two equations

$$V=0 \quad \text{and} \quad A=0.$$

When the proposed equation has three roots equal to a, namely, $a=b=c$, two of the roots of the equation (d) become nothing, at the same time, namely, $b-a$ and $c-a$. In this case the equation (d) will be divisible twice successively by $y-0$ (179) or y; but this can happen, only when the coefficients A and B are nothing; the value of a must then satisfy, at the same time, the three equations

$$V=0, \quad A=0, \quad B=0.$$

Pursuing the same reasoning, we shall perceive, that when the proposed equation has four equal roots, the equation (d) will have three roots equal to zero, or will be divisible three times successively by y; the coefficients A, B and C must then be nothing, at the same time, and consequently the value of a must satisfy at once the four equations

$$V=0, \quad A=0, \quad B=0, \quad C=0.$$

By means of what has been said, we shall not only be able to ascertain, whether a given root is found several times among the roots of the proposed equation, but may deduce a method of determining, whether this equation has roots repeated, of which we are ignorant.

For this purpose, it may be observed, that when we have $A=0$, or

$$ma^{m-1}+(m-1)Pa^{m-2}+(m-2)Qa^{m-3}\ldots+T=0,$$

we may consider a as the root of the equation

$$mx^{m-1}+(m-1)Px^{m-2}+(m-2)Qx^{m-3}\ldots+T=0,$$

x representing, in this case, any unknown quantity whatever; and since a is also the root of the equation $V=0$, or

$$x^m+Px^{m-1}+\text{\&c.}=0,$$

it follows, (189) that $x-a$ is a factor common to the two above mentioned equations.

Changing in the same manner a into x in the quantities B, C, &c. the binomial $x-a$ becomes likewise a factor of the two new equations $B=0$, $C=0$, &c. if the root a reduces to nothing the original quantities B, C, &c.

What has been said with respect to the root a may be applied to every other root, which is several times repeated; thus by seeking, according to the method given for finding the greatest common divisor, the factors common to the equations

$$V=0, \quad A=0, \quad B=0, \quad C=0, \text{\&c.}$$

$y = \alpha$, we shall have, at the same time, $y = -\alpha$. Hence it follows, that this equation must be made up of terms involving only even powers of the unknown quantity; for its first member must be the product of a certain number of factors of the second degree of the form

$$y^2 - \alpha^2 = (y - \alpha)(y + \alpha) \ (184);$$

it will therefore itself be exhibited under the form

$$y^{2n} + p\,y^{2n-2} + q\,y^{2n-4} \ldots\ldots + t\,y^2 + u = 0.$$

If we put $y^2 = z$, this becomes

$$z^n + p\,z^{n-1} + q\,z^{n-2} \ldots\ldots + t\,z + u = 0;$$

and as the unknown quantity z is the square of y, its values will be the squares of the differences between the roots of the proposed equation.

It may be observed, that as the differences between the real roots of the proposed equation are necessarily real, their squares will be positive, and consequently the equation in z will have only positive roots, if the proposed equation admits of those only, which are real.

Let there be, for example, the equation

$$x^3 - 7\,x + 7 = 0;$$

putting $x = a + y$, we have

$$\left.\begin{array}{l} a^3 + 3\,a^2\,y + 3\,a\,y^2 + y^3 \\ -7\,a - 7\,y \\ +7 \end{array}\right\} = 0.$$

Suppressing the terms $a^3 - 7\,a + 7$, which, from their identity with the proposed equation, become nothing when united, and dividing the remainder by y, we have

$$3\,a^2 + 3\,a\,y + y^2 - 7 = 0;$$

eliminating a by means of this equation and the equation

$$a^3 - 7\,a + 7 = 0,$$

we have

$$y^6 - 42\,y^4 + 441\,y^2 - 49 = 0;$$

putting $z = y^2$, this becomes

$$z^3 - 42\,z^2 + 441\,z - 49 = 0.$$

209. The substitution of $a + y$ in the place of x in the equation

$$x^m + P\,x^{m-1} + Q\,x^{m-2} \ldots\ldots + U = 0 \ (204),$$

is sometimes resorted to also in order to make one of the terms of this equation to disappear. We then arrange the result with reference to the powers of y, which takes the place of the unknown quantity x, and consider a as a second unknown quan-

$$5x^4 - 52x^3 + 201x^2 - 342x + 216 = 0;$$

the divisor common to this and the proposed equation is

$$x^3 - 8x^2 + 21x - 18.$$

As this divisor is of the third degree, it must itself contain several factors; we must therefore seek, whether it does not contain some, that are common to the equation $B = 0$, which is here

$$20x^3 - 156x^2 + 402x - 342 = 0.$$

We find, in fact, for a result $x - 3$; the proposed equation then has three roots equal to 3, or admits of $(x-3)^3$ among the number of its factors. Dividing the first common divisor by $x - 3$, as many times as possible, that is, in this case twice, we obtain $x - 2$. As this divisor is common only to the proposed equation, and to the equation $A = 0$, it can enter only twice into the proposed equation. It is evident then, that this equation is equivalent to

$$(x-3)^3 (x-2)^2 = 0.$$

208. As the equation (d) gives the difference between b and the several other roots, when b is substituted for a, the difference between c and the others, when c is substituted for a, &c. and undergoes no change in its form by these several substitutions, retaining the coefficients belonging to the equation proposed, it may be converted into a general equation, which shall give all the differences between the several roots combined two and two. For this purpose, it is only necessary to eliminate a by means of the equation

$$a^m + Pa^{m-1} + Qa^{m-2} \ldots\ldots + Ta + U = 0;$$

for the result being expressed simply by the coefficients, and exhibiting the root under consideration in no form whatever, answers alike to all the roots.

It is evident, that the final equation must be raised to the degree $m(m-1)$; for its roots

$$\begin{array}{llll} a-b, & a-c, & a-d, & \text{\&c.} \\ b-a, & b-c, & b-d, & \text{\&c.} \\ c-a, & c-b, & c-d, & \text{\&c.} \end{array}$$

are equal in number to the number of arrangements, which the m letters a, b, c, &c. admit of when taken two and two. Moreover, since the quantities

$a-b$ and $b-a$, $a-c$ and $c-a$, $b-c$ and $c-b$, &c.

differ only in the sign, the roots of the equation are equal, when taken two and two independently of the signs; so that if we have

It is not difficult to discover the reason of this similarity. By making the last term of the equation in y equal to zero, we suppose, that one of the values of this unknown quantity is zero; and if we admit this supposition with respect to the equation $x = y + a$, it follows that $x = a$; that is, the quantity a, in this case, is necessarily one of the values of x.

210. We have sometimes occasion to resolve equations into factors of the second and higher degrees. I cannot here explain in detail the several processes, which may be employed for this purpose; one example only will be given.

Let there be the equation

$$x^5 - 24x^3 + 12x^2 - 11x + 7 = 0,$$

in which it is required to determine the factors of the third degree; I shall represent one of these factors by

$$x^3 + px^2 + qx + r,$$

the coefficients p, q and r being indeterminate. They must be such, that the first member of the proposed equation will be exactly divisible by the factor

$$x^3 + px^2 + qx + r,$$

independently of any particular value of x; but in making an actual division, we meet with a remainder

$$\begin{aligned} &-(p^3 - 2pq - 24p + r - 12)x^2 \\ &-(p^2q - pr - q^2 - 24q + 11)x \\ &-(p^2r - qr - 24r - 7), \end{aligned}$$

an expression, which must be reduced to nothing independently of x, when we substitute for the letters p, q, and r, the values that answer to the conditions of the question. We have then

$$\begin{aligned} p^3 - 2pq - 24p + r - 12 &= 0 \\ p^2q - pr - q^2 - 24q + 11 &= 0 \\ p^2r - qr - 24r - 7 &= 0. \end{aligned}$$

These three equations furnish us with the means of determining the unknown quantities p, q, and r; and it is to a resolution of these that the proposed question is reduced.

Of the resolution of numerical equations by approximation.

211. Having completed the investigation of commensurable divisors, we must have recourse to the methods of finding roots by approximation, which depend on the following principle;

When we arrive at two quantities which substituted in the place of unknown quantity in an equation, lead to two results with con-y signs, we may infer, that one of the roots of the proposed equa- lies between these two quantities, and is consequently real.

et there be, for example, the equation

$$x^3 - 13x^2 + 7x - 1 = 0;$$

e substitute successively 2 and 20 in the place of x, in the t member, instead of being reduced to zero, this member mes, in the former case, equal to -31, and in the latter, to 939; we may therefore conclude, that this equation has a l root between 2 and 20, that is, greater than two and less n 20.

s there will be frequent occasion to express this relation, I ll employ the signs $>$ and $<$, which algebraists have adopted enote the inequality of two magnitudes, placing the greater wo quantities opposite the opening of the lines, and the less inst the point of meeting. Thus I shall write

$x > 2$, to denote, that x is greater than 2,

$x < 20$, to denote, that x is less than 20.

Tow in order to prove what has been laid down above, we r reason in the following manner. Bringing together the posi- terms of the proposed equation, and also those, which are ative, we have

$$x^3 + 7x - (13x^2 + 1),$$

antity, which will be negative, if we suppose $x = 2$, because n this supposition,

$$x^3 + 7x < 13x^2 + 1,$$

which becomes positive, when we make $x = 20$, because in case

$$x^3 + 7x > 13x^2 + 1.$$

reover it is evident, that the quantities

$$x^3 + 7x \quad \text{and} \quad 13x^2 + 1,$$

h increase, as greater and greater values are assigned to and that, by taking values, which approach each other y nearly, we may make the increments of the proposed ntities as small as we please. But since the first of the above ntities, which was originally less than the second, becomes ater, it is evident, that it increases more rapidly than the er, in consequence of which its deficiency is made up, and it

comes at length to exceed the other; there must therefore be a point, at which the two magnitudes are equal.

The value of x, whatever it be, which renders

$$x^3 + 7x = 13x^2 + 1,$$

and such a value has been proved to exist, gives

$$x^3 + 7x - (13x^2 + 1) = 0,$$

or

$$x^3 - 13x^2 + 7x - 1 = 0,$$

and must necessarily, therefore, be the root of the equation proposed.

What has been shown with respect to the particular equation

$$x^3 - 13x^2 + 7x - 1 = 0,$$

may be affirmed of any equation whatever, the positive terms of which I shall designate by P, and the negative by N. Let a be the value of x, which leads to a negative result, and b that which leads to a positive one; these consequences can take place only upon the supposition, that by substituting the first value, we have $P < N$, and by substituting the second, $P > N$; P therefore from being less, having become greater than N, we conclude as above, that there exists a value of x between a and b, which gives $P = N$.*

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* The above reasoning, though it may be regarded as sufficiently evident, when considered in a general view, has been developed by M. Encontre in a manner, that will be found to be useful to those, who may wish to see the proofs given more in detail.

1. It is evident, that the increments of the polynomials P and N may be made as small as we please. Let

$$P = \alpha x^m + \beta x^n \ldots + \delta x^q,$$

m being the highest exponent of x; if we put $a + y$ in the place of x, this polynomial takes the form

$$A + By + Cy^2 \ldots\ldots + Ty^m,$$

the coefficients $A, B, C, \ldots T$, being finite in number and having a finite value; the first term A will be the value the polynomial P assumes, when $x = a$; the remainder

$$By + Cy^2 \ldots + Ty^m = y(B + Cy \ldots Ty^{m-1})$$

will be the quantity, by which the same polynomial is increased when we augment by y the value $x = a$. This being admitted, if S designate the greatest of the coefficients $B, C, \ldots T$, we have

$$B + Cy \ldots + Ty^{m-1} < S(1 + y \ldots + y^{m-1});$$

now

$$1 + y \ldots + y^{m-1} = \frac{1 - y^m}{1 - y} \ (158);$$

The statement here given seems to require, that the values assigned to x should be both positive or both negative, for if they have different signs, that which is negative produces a change in the signs of those terms of the proposed equation, which contain odd powers of the unknown quantity, and consequently the expressions P and N are not formed in the same manner, when we substitute one value, as when we substitute the other. This difficulty vanishes, if we make $x=0$; in this case, the proposed equation reduces itself to its last term, which has necessarily a sign contrary to that of the result arising from the substitution of one or the other of the above mentioned values. Let there be, for example, the equation

$$x^4-2x^3-3x^2-15x-3=0,$$

the first member of which, when we put

$$x=-1 \quad \text{and} \quad x=2,$$

becomes $+12$ and -45. If we suppose $x=0$, this member is reduced to -3; substituting therefore

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therefore

$$y(B+Cy\ldots.+Ty^{m-1}) < Sy\frac{(1-y^m)}{1-y},$$

and consequently the quantity, by which the polynomial P is increased, will be less than any given quantity m, if we make $\frac{Sy(1-y^m)}{1-y}$ less than this last quantity; this is effected by making $\frac{Sy}{1-y}=m$, because in this case $y=\frac{m}{S+m}$ being <1, the quantity $\frac{Sy(1-y^m)}{1-y}$, equal to $\frac{Sy}{1-y}-\frac{Sy^{m+1}}{1-y}$, will necessarily be less than the quantity m, which is indefinitely small.

2. If we designate by h the increment of the polynomial P and by k, that of the polynomial N, the change, which will be produced in the value of their difference will be $h-k$, and may be rendered smaller than a given quantity by making smaller than this same quantity the increment, which is the greater of the two; we may therefore in the interval between $x=a$ and $x=b$, take values, which shall make the difference of the polynomials P and N change by quantities as small as we please, and since this difference passes in this interval from positive to negative, it may be made to approach as near to zero as we choose. See *Annales de Mathématiques pures et appliquées*, published by M. Gergonne, vol. iv. p. 210.

$$x = 0 \quad \text{and} \quad x = -1,$$

we arrive at two results with contrary signs; but putting $-y$ in the place of x, the proposed equation is changed to

$$y^4 + 2y^3 - 3y^2 + 15y - 3 = 0,$$

and we have

$$P = y^4 + 2y^3 + 15y, \quad N = 3y^2 + 3,$$

whence

$$P < N, \text{ when } y = 0,$$
$$P > N, \text{ when } y = 1.$$

Reasoning as before, we may conclude, that the equation in y has a real root found between 0 and $+1$; whence it follows, that the root of the equation in x lies between 0 and -1, and consequently between $+2$ and -1.

As every case the proposition enunciated can present, may be reduced to one or the other of those, which have been examined, the truth of this proposition is sufficiently established.

212. Before proceeding further, I shall observe, that *whatever be the degree of an equation, and whatever its coefficients, we may always assign a number, which substituted for the unknown quantity will render the first term greater than the sum of all the others.* The truth of this proposition will be immediately apparent from what has been intimated of the rapidity, with which the several powers of a number greater than unity increase (126); since the highest of these powers exceeds those below it more and more in proportion to the increased magnitude of the number employed, so that there is no limit to the excess of the first above each of the others. Observe moreover the method, by which we may find a number that fulfils the condition required by the enunciation.

It is evident, that the case most unfavourable to the supposition is that, in which we make all the coefficients of the equation negative, and each equal to the greatest, that is, when instead of

$$x^m + Px^{m-1} + Qx^{m-2} \ldots\ldots + Tx + U = 0,$$

we take

$$x^m - Sx^{m-1} - Sx^{m-2} \ldots\ldots - Sx - S = 0,$$

S representing the greatest of the coefficients P, Q, ... T, U. Giving to the first member of this equation the form

$$x^m - S(x^{m-1} + x^{m-2} \ldots\ldots + 1),$$

we may observe, that

$$x^{m-1} + x^{m-2} \ldots\ldots + 1 = \frac{x^m - 1}{x - 1} \text{ (158);}$$

eceding expression then may be changed into

$$x^m - \frac{S(x^m - 1)}{x - 1}, \text{ or into } x^m - \frac{Sx^m}{x-1} + \frac{S}{x-1}.$$

substitute M for x, this becomes

$$M^m - \frac{SM^m}{M-1} + \frac{S}{M-1},$$

ntity, which evidently becomes positive, if we make

$$M^m = \frac{SM^m}{M-1}.$$

if we divide each member of this equation by M^m, we have

$$1 = \frac{S}{M-1} \quad \text{or} \quad M = S + 1.$$

ibstituting therefore for x the greatest of the coefficients in the equation, augmented by unity, we render the first greater than the sum of all the others.

smaller number may be taken for M, if we wish simply nder the positive part of the equation greater than the nega- for to do this it is only necessary to render the first term er than the sum arising from all the others, when their cients are each equal not to the greatest among all the cients, but to the greatest of those, which are negative; we therefore merely to take for M this coefficient augmented by .*

nce it follows, that the positive roots of the proposed equa- are necessarily comprehended within 0 and $S + 1$.

the same way we may discover a limit to the negative roots; his purpose we must substitute $-y$ for x, in the proposed tion, and render the first term positive, if it becomes nega- (178). It is evident, that by a transformation of this kind, ositive values of y answer to the negative values of x, and the rse. If R be the greatest negative coefficient after this ge, $R + 1$ will form a limit to the positive values of y; con- ently $-R - 1$ will form that of the negative values of x.

astly, if we would find for the smallest of the roots a limit oaching as near to zero as possible, we may arrive at it by

* In the *Résolution des équations numériques*, by Lagrange, there formulas, which reduce this number to narrower limits, but has been said above is sufficient to render the fundamental ositions for the resolution of numerical equations independent of consideration of infinity.

substituting $\frac{1}{y}$ for x in the proposed equation, and preparing the equation in y, which is thus obtained, according to the directions given in art. 178. As the values of y are the reverse of those of x, the greatest of the first will correspond to the least of the second, and reciprocally the greatest of the second to the least of the first. If therefore $S'+1$ represent the highest limit to the values of y, that is, if

$$y < S'+1,$$

which gives

$$\frac{1}{x} < S'+1,$$

we shall have successively

$$1 < (S'+1)\,x,\ \frac{1}{S'+1} < x.$$

Indeed, it is very evident, that we may, without altering the relative magnitude of two quantities separated by the sign $<$ or $>$, multiply or divide them by the same quantity, and that we may also add the same quantity to or subtract it from each side of the signs $<$ and $>$, which possess, in this respect, the same properties as the sign of equality.

213. It follows from what precedes, that *every equation of a degree denoted by an odd number has necessarily a real root affected with a sign contrary to that of its last term*; for if we take the number M such, that the sign of the quantity

$$M^m + PM^{m-1} + QM^{m-2} \ldots\ldots + TM \pm U$$

depends solely on that of its first term M^m, the exponent m being an odd number, the term M^m will have the same sign as the number M (128). This being admitted, if the last term U has the sign $+$, and we make $x = -M$, we shall arrive at a result affected with a sign contrary to that, which the supposition of $x = 0$ would give; from which it is evident, that the proposed equation has a root between 0 and $-M$, that is, a negative root. If the last term U has the sign $-$, we make $x = +M$; the result will then have a sign contrary to that given by the supposition of $x = 0$, and in this case, the root will be found between 0 and $+M$, that is, it will be positive.

214. When the proposed equation is of a degree denoted by an even number, as the first term M^m remains positive, whatever sign we give to M, we are not, by the preceding observations, furnished with the means of proving the existence of a real root.

if the last term has the sign +, since, whether we make $x=0$, or $x=\pm M$, we have always a positive result. But when this term is negative, we find, by making

$$x=+M, \quad x=0, \quad x=-M,$$

three results affected respectively with the signs +, — and +, and consequently the proposed equation has, at least, two real roots in this case, the one positive found between M and 0, the other negative between 0 and $-M$; therefore *every equation of an even degree, the last term of which is negative, has at least two real roots, the one positive and the other negative.*

215. I now proceed to the resolution of equations by approximation; and in order to render what is to be offered on this subject more clear, I shall begin with an example.

Let there be the equation

$$x^4-4x^3-3x+27=0;$$

the greatest negative coefficient found in this equation being -4, it follows (212), that the greatest positive root will be less than 5. Substituting $-y$ for x we have

$$y^4+4y^3+3y+27=0;$$

and as all the terms of this result are positive, it appears, that y must be negative; whence it follows, that x is necessarily positive, and that the proposed equation can have no negative roots; its real roots are therefore found between 0 and +5.

The first method, which presents itself for reducing the limits, between which the roots are to be sought, is to suppose successively

$$x=1, \quad x=2, \quad x=3, \quad x=4;$$

and if two of these numbers substituted in the proposed equation, lead to results with contrary signs, they will form new limits to the roots. Now if we make

$x=1$, the first member of the equation becomes $+21$,
$x=2$ . . . . $+5$,
$x=3$ . . . . -9,
$x=4$ . . . . $+15$;

it is evident therefore, that this equation has two real roots, the one found between 2 and 3, and the other between 3 and 4. To approximate the first still nearer, we take the number 2,5, which occupies the middle place between 2 and 3 (*Arith.* 129), the present limits of this root; making then $x=2,5$, we arrive at the result

$$+39{,}0625-62{,}5-7{,}5+27=-3{,}9375\,;$$

as this result is negative, it is evident, that the root sought is between 2 and 2,5. The mean of these two numbers is 2,25; taking $x=2{,}3$ we have the root sought within about one tenth of its value, and shall approximate the true root very fast by the following process, given by Newton.

We make $x=2{,}3+y$; it is evident, that the unknown quantity y amounts only to a very small fraction, the square and higher powers of which may be neglected; we have then

$$\begin{aligned} x^4 &= (2{,}3)^4 + 4\,(2{,}3)^3\,y \\ -4\,x^3 &= -4\,(2{,}3)^3 - 12\,(2{,}3)^2\,y \\ -3\,x &= -3\,(2{,}3) - 3\,y\,; \end{aligned}$$

substituting these values, the proposed equation becomes

$$-0{,}5839-17{,}812\,y=0,$$

which gives

$$y=-\frac{0{,}5839}{17{,}812}.$$

Stopping at hundredths, we obtain for the result of the first operation

$$y=-0{,}03 \text{ and } x=2{,}3-0{,}03=2{,}27.$$

To obtain a new value of x more exact than the preceding, we suppose $x=2{,}27+y'$; substituting this value in the proposed equation and neglecting all the powers of y' exceeding the first, we find

$$-0{,}04595359-18{,}046468y'=0,$$

whence

$$y'=-\frac{0{,}04595359}{18{,}046468}=-0{,}0025,$$

and consequently $x=2{,}2675$. We may, by pursuing this process, approximate, as nearly as we please, the true value of x.

If we seek the second root, contained between 3 and 4, by the same method, we find, stopping at the fourth decimal place,

$$x=3{,}6797.$$

216. We may ascertain the exactness of the method above explained, by seeking the limit to the values of the terms, which are neglected.

If the proposed equation were

$$x^m+P\,x^{m-1}+Q\,x^{m-2}\,\ldots\ldots\ldots+T\,x+U=0,$$

substituting $a+y$ for x, we should have for the result the first of the equations found in art. 204, because a being not the root of

the equation, but only an approximate value of x, cannot reduce to nothing the quantity

$$a^m + P a^{m-1} + Q a^{m-2} \ldots\ldots + T a + U.$$

Representing this last by V, we have, instead of the equation (d) above referred to, the following

$$V + \frac{A}{1} y + \frac{B}{1.2} y^2 + \frac{C}{1.2.3} y^3 \ldots\ldots + y^m = 0,$$

from which we obtain

$$A y = - V - \frac{B}{1.2} y^2 - \frac{C}{1.2.3} y^3 \ldots\ldots - y^m,$$

$$y = - \frac{V}{A} - \frac{B y^2}{1.2 A} - \frac{C y^3}{1.2.3 A} \ldots\ldots - \frac{y^m}{A}.$$

Neglecting the powers of y exceeding the first, we have

$$y = - \frac{V}{A},$$

and this value differs from the real value of y by

$$- \frac{B y^2}{1.2 A} - \frac{C y^3}{1.2.3 A} \ldots\ldots - \frac{y^m}{A}.$$

If a differs from the true value of x only by a quantity less than $\frac{1}{p} a$, the above mentioned error becomes less than that, which would arise from putting $\frac{1}{p} a$ in the place of y, which would give

$$- \frac{B}{1.2 A} \left(\frac{a}{p}\right)^2 - \frac{C}{1.2.3 A} \left(\frac{a}{p}\right)^3 \ldots\ldots - \frac{1}{A}\left(\frac{a}{p}\right)^m.$$

Finding the value of this quantity, we shall be able to determine, whether it may be neglected when considered with reference to $\frac{V}{A}$, and if it be found too large, we must obtain for a a number, which approaches nearer to the true value of x.

To conclude, when we have gone through the calculation with several numbers y, y', y'', &c. if the results thus obtained, form a decreasing series, an approximation is certain.

217. The method we have employed above, is called the *Method by successive Substitutions*. Lagrange has considerably improved it.* He has remarked, that by substituting only

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* See *Résolution des Equations numériques.*

entire numbers, we may pass over several roots without perceiving them. In fact, if we have, for example, the equation

$$(x-\tfrac{1}{3})\,(x-\tfrac{1}{2})\,(x-3)\,(x-4)=0,$$

by substituting for x the numbers 0, 1, 2, 3, &c. we shall pass over the roots $\frac{1}{3}$ and $\frac{1}{2}$, without discovering, that they exist; for we shall have

$$(0-\tfrac{1}{3})\,(0-\tfrac{1}{2})\,(0-3)\,(0-4)=+\tfrac{1}{3}\times\tfrac{1}{2}\times 3\times 4$$

$$(1-\tfrac{1}{3})\,(1-\tfrac{1}{2})\,(1-3)\,(1-4)=+\tfrac{2}{3}\times\tfrac{1}{2}\times 2\times 3,$$

results affected by the same sign. It will be readily perceived, that this circumstance takes place in consequence of the fact, that the substitution of 1 for x changes at the same time the signs of both the factors $x-\frac{1}{3}$ and $x-\frac{1}{2}$, which pass from the negative state, in which they are when 0 is put in the place of x, to the positive; but if we substitute for x a number between $\frac{1}{3}$ and $\frac{1}{2}$, the sign of the factor $x-\frac{1}{3}$ alone will be changed, and we shall obtain a negative result.

We shall necessarily meet with such a number, if we substitute, in the place of x, numbers, which differ from each other by a quantity less, than the difference between the roots $\frac{1}{3}$ and $\frac{1}{2}$. If, for example, we substitute $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, &c. there will be two changes of the sign.

It may be objected to the above example, that when the fractional coefficients of an equation have been made to disappear, the equation can have for roots only either entire or irrational numbers, and not fractions; but it will be readily seen, that the irrational numbers, for which we have, in the example, substituted fractions for the purpose of simplifying the expressions, may differ from each other by a quantity less than unity.

In general, the results will have the same sign, whenever the substitutions produce a change in the sign of an even number of factors.* To obviate this inconvenience we must take the numbers to be substituted, such, that the difference between the smallest limit and the greatest, will be less than the least of the differences, which can exist between the roots of the proposed equation; by this means the numbers to be substituted will necessarily fall

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* Equal roots cannot be discovered by this process, when their number is even; to find these we must employ the method given in art. 205.

between the successive roots, and will cause a change in the sign of one factor only. This process does not presuppose the smallest difference between the roots to be known, but requires only that the limit, below which it cannot fall, be determined.

In order to obtain this limit, we form the equation involving the squares of the differences of the roots (208).

Let there be the equation

$$z^n + p z^{n-1} + q z^{n-2} \dots + t z + u = 0 \dots (D),$$

to obtain the smallest limit to the roots, we make (212) $z = \frac{1}{v}$; we have then the equation

$$\frac{1}{v^n} + p \frac{1}{v^{n-1}} + q \frac{1}{v^{n-2}} \dots + t \frac{1}{v} + u = 0,$$

or, reducing all the terms to the same denominator,

$$1 + p v + q v^2 \dots\dots + t v^{n-1} + u v^n = 0,$$

then disengaging v^n,

$$v^n + \frac{t}{u} v^{n-1} \dots + \frac{q}{u} v^2 + \frac{p}{u} v + \frac{1}{u} = 0;$$

and if $\frac{r}{u}$ represent the greatest negative coefficient found in this equation, we shall have

$$\frac{1}{\frac{r}{u} + 1} < z.$$

It is only necessary to consider here the positive limit, as this alone relates to the real roots of the proposed equation.

Knowing the limit

$$\frac{1}{\frac{r}{u} + 1} = \frac{u}{r + u},$$

less than the square of the smallest difference between the roots of the proposed equation, we may find its square root, or at least, take the rational number next below this root; this number, which I shall designate by k, will represent the difference, which must exist between the several numbers to be substituted. We thus form the two series,

$$0, +k, +2k, +3k, \text{ \&c.}$$
$$-k, -2k, -3k, \text{ \&c.}$$

from which we are to take only the terms, comprehended between the limits to the smallest and the greatest positive roots, and those to the smallest and the greatest negative roots of the

proposed equation. Substituting these different numbers, we shall arrive at a series of results, which will show by the changes of the sign that take place, the several real roots, whether positive or negative.

218. Let there be, for example, the equation

$$x^3 - 7x + 7 = 0,$$

from which in art. 208 was derived the equation

$$z^3 - 42z^2 + 441z - 49 = 0;$$

making $z = \frac{1}{v}$, and, after substituting this value, arranging the result with reference to v, we have

$$v^3 - 9v^2 + \tfrac{42}{49}v - \tfrac{1}{49} = 0,$$

from which we obtain

$$v < 10, \ z > \tfrac{1}{10};$$

we must therefore take $k =$ or $< \frac{1}{\sqrt{10}}$. This condition will be fulfilled, if we make $k = \frac{1}{4}$; but it is only necessary to suppose $k = \frac{1}{3}$; for by putting 9 in the place of v in the preceding equation, we obtain a positive result, which must become greater, when a greater value is assigned to v, since the terms v^3 and $9v^2$ already destroy each other, and $\frac{42}{49}v$ exceeds $\frac{1}{49}$.

The highest limit to the positive roots of the proposed equation

$$x^3 - 7x + 7 = 0,$$

is 8, and that to the negative roots -8; we must therefore substitute for x the numbers

$$0, \ \tfrac{1}{3}, \ \tfrac{2}{3}, \ \tfrac{3}{3}, \ \tfrac{4}{3}, \ldots\ldots \ \tfrac{24}{3},$$
$$-\tfrac{1}{3}, -\tfrac{2}{3}, -\tfrac{3}{3}, -\tfrac{4}{3}, \ldots\ldots -\tfrac{24}{3}.$$

We may avoid fractions by making $x = \frac{x'}{3}$; for in this case the differences between the several values of x' will be triple of those between the values of x, and consequently will exceed unity; we shall then have only to substitute successively

$$0, \ 1, \ 2, \ 3, \ldots\ldots \ 24,$$
$$-1, -2, -3, \ldots\ldots -24,$$

in the equation

$$x'^3 - 63x' + 189 = 0.$$

The signs of the results will be changed between $+4$ and $+5$, between $+5$ and $+6$, and between -9 and -10, so that we shall have for the positive values,

$$\left.\begin{array}{l} x' > 4 \text{ and } < 5 \\ x' > 5 \text{ and } < 6 \end{array}\right\} \text{whence} \left\{\begin{array}{l} x > \frac{4}{3} \text{ and } < \frac{5}{3} \\ x > \frac{5}{3} \text{ and } < \frac{6}{3} \end{array}\right.$$

and the negative value of x' will be found between -9 and -10, that of x between $-\frac{9}{3}$ and $-\frac{10}{3}$.

Knowing now the several roots of the proposed equation within $\frac{1}{3}$, we may approach nearer to the true value by the method explained in art. 215.

219. The methods employed in the example given in art. 215, and in the preceding article, may be applied to an equation of any degree whatever, and will lead to values approaching the several real roots of this equation. It must be admitted however, that the operation becomes very tedious, when the degree of the proposed equation is very elevated; but in most cases it will be unnecessary to resort to the equation (D), or rather its place may be supplied by methods, with which the study of the higher branches of analysis will make us acquainted.*

I shall observe however, that by substituting successively the numbers 0, 1, 2, 3, &c. in the place of x, we shall often be lead to suspect the existence of roots, that differ from each other by a quantity less than unity. In the example, upon which we have been employed, the results are

$$+7, \ +1, \ +1, \ +13,$$

which begin to increase after having decreased from $+7$ to $+1$. From this order being reversed it may be supposed, that between the numbers $+1$ and $+2$ there are two roots either equal or nearly equal. To verify this supposition, the unknown quantity should be multiplied. Making $x = \frac{y}{10}$, we find

$$y^3 - 700y + 7000 = 0,$$

an equation, which has two positive roots, one between 13 and 14, and the other between 16 and 17.

The number of trials necessary for discovering these roots is not great; for it is only between 10 and 20, that we are to search for y; and the values of this unknown quantity being

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* A very elegant method, given by Lagrange for avoiding the use of the equation (D), may be found in the *Traité de la Résolution des Equations numériques.*

determined in whole numbers, we may find those of x within one tenth of unity.

220. When the coefficients in the equation proposed for resolution are very large, it will be found convenient to transform this equation into another, in which the coefficients shall be reduced to smaller numbers. If we have, for example,

$$x^4 - 80\,x^3 + 1998\,x^2 - 14937\,x + 5000 = 0,$$

we may make $x = 10\,z$; the equation then becomes

$$z^4 - 8\,z^3 + 19{,}98\,z^2 - 14{,}937\,z + 0{,}5 = 0.$$

If we take the entire numbers, which approach nearest to the coefficients in this result, we shall have

$$z^4 - 8\,z^3 + 20\,z^2 - 15\,z + 0{,}5 = 0.$$

It may be readily dicovered, that z has two real values, one between 0 and 1, the other between 1 and 2, whence it follows, that those of the proposed equation are between 0 and 10, and 10 and 20.

I shall not here enter into the investigation of imaginary roots, as it depends on principles we cannot at present stop to illustrate; I shall pursue the subject in the *Supplement*.

221. Lagrange has given to the successive substitutions a form, which has this advantage, that it shows immediately, what approaches we make to the true root by each of the several operations, and which does not presuppose the value to be known within one tenth.

Let a represent the entire number immediately below the root sought; to obtain this root, it will be only necessary to augment a by a fraction; we have therefore $x = a + \frac{1}{y}$. The equation involving y, with which we are furnished by substituting this value in the proposed equation, will necessarily have one root greater than unity; taking b to represent the entire number immediately below this root, we have for the second approximation $x = a + \frac{1}{b}$. But b having the same relation to y, which a has to x, we may, in the equation involving y, make $y = b + \frac{1}{y'}$, and y' will necessarily be greater than unity; representing by b' the entire number immediately below the root of the equation in y', we have

$$y = b + \frac{1}{b'} = \frac{b\,b' + 1}{b'}$$

substituting this value in the expression for x, we have

$$x = a + \frac{b'}{b\,b' + 1},$$

for the third approximation to x. We may find a fourth by making $y' = b' + \frac{1}{y''}$; for if b'' designate the entire number immediately below y'', we shall have

$$y' = b' + \frac{1}{b''} = \frac{b'\,b'' + 1}{b''},$$

whence

$$y = b + \frac{b''}{b'\,b'' + 1} = \frac{b\,b'\,b'' + b'' + b}{b'\,b'' + 1},$$

$$x = a + \frac{b'\,b'' + 1}{b\,b'\,b'' + b'' + b},$$

and so on.

222. I shall apply this method to the equation

$$x^3 - 7x + 7 = 0.$$

We have already seen (218), that the smallest of the positive roots of this equation is found between $\frac{4}{3}$ and $\frac{5}{3}$, that is, between 1 and 2; we make therefore $x = 1 + \frac{1}{y}$; we shall then have

$$y^3 - 4y^2 + 3y + 1 = 0.$$

The limit to the positive roots of this last equation is 5, and by substituting successively 0, 1, 2, 3, 4, in the place of y, we immediately discover, that this equation has two roots greater than unity, one between 1 and 2, and the other between 2 and 3. Hence

$$x = 1 + \tfrac{1}{1} \quad \text{and} \quad x = 1 + \tfrac{1}{2},$$

that is

$$x = 2 \quad \text{and} \quad x = \tfrac{3}{2}.$$

These two values correspond to those, which were found above between $\frac{6}{3}$ and $\frac{5}{3}$, and between $\frac{5}{3}$ and $\frac{4}{3}$, and which differ from each other by a quantity less than unity.

In order to obtain the first, which answers to the supposition of $y = 1$, to a greater degree of exactness, we make

$$y = 1 + \frac{1}{y'},$$

we then have

$$y'^3 - 2y'^2 - y' + 1 = 0.$$

We find in this equation only one root greater than unity, and that is between 2 and 3, which gives

$$y = 1 + \tfrac{1}{2} = \tfrac{3}{2},$$

whence

$$x = 1 + \tfrac{2}{3} = \tfrac{5}{3}.$$

Again, if we suppose $y' = 2 + \dfrac{1}{y''}$, we shall be furnished with the equation

$$y''^3 - 3\,y''^2 - 4\,y'' - 1 = 0\,;$$

we find the value of y'' to be between 4 and 5; taking the smallest of these numbers 4, we have

$$y' = 2 + \tfrac{1}{4},\quad y = 1 + \tfrac{4}{9} = \tfrac{13}{9},\quad x = 1 + \tfrac{9}{13} = \tfrac{22}{13}.$$

It would be easy to pursue this process, by making $y'' = 4 + \dfrac{1}{y'''}$, and so on.

I return now to the second value of x, which, by the first approximation, was found equal to $\frac{3}{2}$, and which answers to the supposition of $y = 2$. Making $y = 2 + \dfrac{1}{y'}$, and substituting this expression in the equation involving y, we have, after changing the signs in order to render the first term positive,

$$y'^3 + y'^2 - 2\,y' - 1 = 0.$$

This equation, like the corresponding one in the above operation, has only one root greater than unity, which is found between 1 and 2; taking $y' = 1$, we have

$$y = 3,\qquad x = \tfrac{4}{3}.$$

Again assuming

$$y' = 1 + \frac{1}{y''},$$

we are furnished with the equation

$$y''^3 - 3\,y''^2 - 4\,y'' - 1 = 0,$$

in which y'' is found to be between 4 and 5, whence

$$y' = \tfrac{5}{4},\qquad y = \tfrac{14}{5},\qquad x = \tfrac{19}{14}.$$

We may continue the process by making $y'' = 4 + \dfrac{1}{y'''}$, and so on.

The equation $x^3 - 7\,x + 7 = 0$ has also one negative root between -3 and -4. In order to approach it more nearly, we make $x = -3 - \dfrac{1}{y}$; which gives

$$y^3 - 20\,y^2 - 9\,y - 1 = 0,\quad y > 20 \text{ and } < 21,$$

whence

$$x = -3 - \tfrac{1}{20} = -\tfrac{61}{20}.$$

To proceed further, we may suppose $y = 20 + \frac{1}{y'}$, &c. we shall then obtain successively values more and more exact.

The several equations transformed into equations in y, y', y'', &c. will have only one root greater than unity, unless two or more roots of the proposed equation are comprehended within the same limits a and $a + 1$; when this is the case, as in the above example, we shall find in some one of the equations in y, y', &c. several values greater than unity. These values will introduce the different series of equations, which show the several roots of the proposed equation, that exist within the limits a and $a + 1$.

The learner may exercise himself upon the following equation

$$x^3 - 2x - 5 = 0,$$

the real root of which is between 2 and 3; we find for the entire values of y, y', &c.

$$10,\ 1,\ 1,\ 2,\ 1,\ 3,\ 1,\ 1,\ 12,\ \&c.$$

and for the approximate values of x,

$$\frac{2}{1},\ \frac{21}{10},\ \frac{23}{11},\ \frac{44}{21},\ \frac{111}{53},\ \frac{155}{74},\ \frac{576}{275},\ \frac{731}{349},\ \frac{1307}{624},\ \frac{16415}{7837}.$$

Of proportion and progression.

223. Arithmetic introduces us to a knowledge of the definition and fundamental properties of *proportion* and *equidifference*, or of what is termed *geometrical* and *arithmetical proportion.* I now proceed to treat of the application of algebra to the principles there developed; this will lead to several results, of which frequent use is made in geometry.

I shall begin by observing, that equidifference and proportion may be expressed by equations. Let A, B, C, D, be the four terms of the former, and a, b, c, d, the four terms of the latter; we have then

$$B - A = D - C \text{ (\textit{Arith.} 127)}, \quad \frac{b}{a} = \frac{d}{c} \text{ (\textit{Arith.} 111)},$$

equations, which are to be regarded as equivalent to the expressions

$$A \,.\, B : C \,.\, D, \qquad a : b :: c : d,$$

and which give

$$A + D = B + C, \qquad ad = bc.$$

Hence it follows, that *in equidifference the sum of the extreme terms is equal to that of the means*, and *in proportion the product*

of the extremes is equal to the product of the means, as has been shown in Arithmetic (127, 113), by reasonings, of which the above equations are only a translation into algebraic expressions.

The reciprocal of each of the preceding propositions may be easily demonstrated; for from the equations

$$A + D = B + C, \qquad a\,d = b\,c,$$

we return at once to

$$D - C = B - A, \qquad \frac{b}{a} = \frac{d}{c},$$

and consequently, *when four quantities are such, that two among them give the same sum, or the same product, as the other two, the first are the means and the second the extremes* (or the converse) *of an equidifference or proportion.*

When $B = C$, the equidifference is said to be *continued*; the same is said of proportion, when $b = c$. We have in this case

$$A + D = 2\,B, \qquad a\,d = b^2:$$

that is, *in continued equidifference the sum of the extremes is equal to double the mean*; and *in proportion, the product of the extremes is equal to the square of the mean.* From this we deduce

$$B = \frac{A + D}{2}, \qquad b = \sqrt{a\,d};$$

the quantity B is the *middle* or mean arithmetical proportional between A and D, and the quantity b the *mean geometrical proportional* between a and d.

The fundamental equations

$$B - A = D - C, \qquad \frac{b}{a} = \frac{d}{c},$$

lead also to the following;

$$C - A = D - B, \qquad \frac{c}{a} = \frac{d}{b};$$

from which it is evident, that we may change the relative places of the means in the expressions $A \,.\, B : C \,.\, D$, $a : b :: c : d$, and in this way obtain $A \,.\, C : B \,.\, D$, $a : c :: b : d$. In general, we may make any transposition of the terms, which is consistent with the equations

$$A + D = B + C \text{ and } a\,d = b\,c \text{ (*Arith.* 114.)}$$

I have now done with equidifference, and shall proceed to consider proportion simply.

224. It is evident, that to the two members of the equation $\frac{b}{a} = \frac{d}{c}$ we may add the same quantity m, or subtract it from them; so that we have

$$\frac{b}{a} \pm m = \frac{d}{c} \pm m;$$

reducing the terms of each member to the same denominator, we obtain

$$\frac{b \pm m a}{a} = \frac{d \pm m c}{c},$$

an equation, which may assume the form

$$\frac{c}{a} = \frac{d \pm m c}{b \pm m a},$$

and may be reduced to the following proportion,

$$b \pm m a : d \pm m c :: a : c;$$

and as $\frac{c}{a} = \frac{d}{b}$, we have likewise

$$\frac{d \pm m c}{b \pm m a} = \frac{d}{b}$$

or

$$b \pm m a : d \pm m c :: b : d.$$

These two proportions may be enunciated thus; *The first consequent plus or minus its antecedent taken a given number of times, is to the second consequent plus or minus its antecedent taken the same number of times, as the first term is to the third, or as the second is to the fourth.*

Taking the sums separately and comparing them together and also the differences, we obtain

$$\frac{d + m c}{b + m a} = \frac{c}{a}, \qquad \frac{d - m c}{b - m a} = \frac{c}{a},$$

whence we conclude

$$\frac{d + m c}{b + m a} = \frac{d - m c}{b - m a},$$

that is

$$b + m a : d + m c :: b - m a : d - m c;$$

or rather, by changing the relative places of the means

$$b + m a : b - m a :: d + m c : d - m c;$$

and if we make $m = 1$, we have simply

$$b + a : b - a :: d + c : d - c,$$

which may be enunciated thus;

The sum of the two first terms is to their difference as the sum of the two last is to their difference.

225. The proportion $a : b :: c : d$ may be written thus;

$$a : c :: b : d;$$

we have then

$$\frac{c}{a} \pm m = \frac{d}{b} \pm m,$$

whence

$$\frac{c \pm m\,a}{a} = \frac{d \pm m\,b}{b},$$

and lastly

$$c \pm m\,a : d \pm m\,b :: a : b \quad \text{or} :: c : d,$$

from which it follows, that *the second antecedent plus or minus the first taken a given number of times, is to the second consequent plus or minus the first taken the same number of times, as any one of the antecedents whatever is to its consequent.*

This proposition may also be deduced immediately from that given in the preceding article; for by changing the order of the means in the original proportion

$$a : b :: c : d,$$

and applying the proposition referred to, we obtain successively

$$a : c :: b : d,$$

$$c \pm m\,a : d \pm m\,b :: a : b \quad \text{or} \quad :: c : d,$$

and denominating the letters a, b, c, d, in this last proportion, according to the place they occupy in the original proportion, we may adopt the preceding enunciation.

Making $m = 1$, we obtain the proportions

$$c \pm a : d \pm b :: a : b$$
$$:: c : d,$$

$$c + a : c - a :: d + b : d - b;$$

whence it appears, that *the sum or difference of the antecedents is to the sum or difference of the consequents, as one antecedent is to its consequent,* and that *the sum of the antecedents is to their difference as that of the consequents is to their difference.*

In general, if we have

$$\frac{b}{a} = \frac{d}{c} = \frac{f}{e} = \frac{h}{g}, \text{ \&c.}$$

and make $\frac{b}{a} = q$, we shall have

$$\frac{d}{c} = q, \quad \frac{f}{e} = q, \quad \frac{h}{g} = q, \text{ \&c.}$$

which gives

$$b = a\,q, \quad d = c\,q, \quad f = e\,q, \quad h = g\,q, \text{ \&c.}$$

then by adding these equations member to member, we obtain

$$b+d+f+h=aq+cq+eq+gq$$

or
$$b+d+f+h=q(a+c+e+g),$$

whence it follows that

$$\frac{b+d+f+h}{a+c+e+g}=q=\frac{b}{a}.$$

This result is enunciated thus; *in a series of equal ratios,*

$$a:b::c:d::e:f::g:h, \text{ \&c.}$$

the sum of any number whatever of antecedents is to the sum of a like number of consequents, as one antecedent is to its consequent.

226. If we have the two equations

$$\frac{b}{a}=\frac{d}{c}, \quad \text{and} \quad \frac{f}{e}=\frac{h}{g},$$

and multiply the first members together and the second together, the result will be

$$\frac{bf}{ae}=\frac{dh}{cg};$$

an equation equivalent to the proportion

$$ae:bf::cg:dh,$$

which may be obtained also by multiplying the several terms of the proportion

$$a:b::c:d,$$

by the corresponding ones in the proportion

$$e:f::g:h.$$

Two proportions multiplied thus term by term are said to be *multiplied in order*; and the products obtained in this way, are, as will be seen, proportional; the new ratios are the ratios *compounded* of the original ratios (*Arith.* 123).

It will be readily perceived also, that if we divide two proportions term by term, or in *order*, the result will be a proportion.

227. If we have
$$\frac{b}{a}=\frac{d}{c},$$

we may deduce from it

$$\frac{b^m}{a^m}=\frac{d^m}{c^m},$$

which gives

$$a^m:b^m::c^m:d^m;$$

whence it follows, *that the squares, the cubes, and in general the similar powers of four proportional quantities are also proportional.*

The same may be said of fractional powers, for since

$$\sqrt[m]{\frac{b}{a}} = \frac{\sqrt[m]{b}}{\sqrt[m]{a}},$$

and

$$\sqrt[m]{\frac{d}{c}} = \frac{\sqrt[m]{d}}{\sqrt[m]{c}};$$

therefore

$$\frac{\sqrt[m]{b}}{\sqrt[m]{a}} = \frac{\sqrt[m]{d}}{\sqrt[m]{c}},$$

or

$$\sqrt[m]{a} : \sqrt[m]{b} :: \sqrt[m]{c} : \sqrt[m]{d},$$

if $a : b :: c : d$; that is, *the roots of the same degree of four proportional quantities; are also proportional.*

Such are the leading principles in the theory of proportion. This theory was invented for the purpose of discovering certain quantities by comparing them with others. Latin names were for a long time used to express the different changes or transformations, which a proportion admits of. We are beginning to relieve the memory of the mathematical student from so unnecessary a burden; and this parade of proportions might be entirely superseded by substituting the corresponding equations, which would give greater uniformity to our methods, and more precision to our ideas.

228. We pass from proportion to progression by an easy transition. After we have acquired the notion of three quantities in continued equidifference, the last of which exceeds the second, as much as this exceeds the first, we shall be able without difficulty to represent to ourselves an indefinite number of quantities a, b, c, d, &c. such, that each shall exceed the preceding one, by the same quantity δ, so that

$$b = a + \delta,\ c = b + \delta,\ d = c + \delta,\ e = d + \delta,\ \&c.$$

A series of these quantities is written thus;

$$\div a . b . c . d . e . f. \&c.$$

and is termed an *arithmetical progression*; I have thought it proper however to change this denomination to that of *progression by differences*. (See *Arith.* art. 127, note.)

We may determine any term whatever of this progression without employing the intermediate ones. In fact, if we substitute for b its value in the expression for c, we have

$$c = a + 2\delta;$$

by means of this last we find

$$d = a + 3\delta, \text{ then } e = a + 4\delta,$$

and so on; whence it is evident, that representing by l the term, the place of which is denoted by n, we have

$$l = a + (n - 1)\delta.$$

Let there be, for example, the progression

$$\div 3 . 5 . 7 . 9 . 11 . 13 . 15 . 17, \&c.$$

here the first term $a = 3$, the difference or *ratio* $\delta = 2$; we find for the eighth term,

$$3 + (8 - 1) 2 = 17,$$

the same result, to which we arrive by calculating the several preceding terms.

The progression we have been considering is called *increasing*; by reversing the order, in which the terms are written, thus,

$$\div 17 . 15 . 13 . 11 . 9 . 7 . 5 . 3 . 1 . -1 . -3, \&c.$$

we form a *decreasing* progression. We may still find any term whatever by means of the formula $a + (n - 1)\delta$, observing only, that δ is to be considered as negative, since in this case we must subtract the difference from any particular term in order to obtain the following.

229. We may also, by a very simple process, determine the sum of any number whatever of terms in a progression by differences. This progression being represented by

$$\div a . b . c \ldots\ldots\ldots\ldots\ldots i . k . l,$$

and S denoting the sum of all the terms, we have

$$S = a + b + c \ldots\ldots\ldots + i + k + l.$$

Reversing the order, in which the terms of the second member of this equation are written, we have still

$$S = l + k + i \ldots\ldots\ldots\ldots + c + b + a.$$

If we add together these equations, and unite the corresponding terms, we obtain

$$2S = (a+l) + (b+k) + (c+i) \ldots + (i+c) + (k+b) + (l+a);$$

but by the nature of the progression, we have, beginning with the first term,

$$a + \delta = b, \; b + \delta = c, \ldots\ldots i + \delta = k, \; k + \delta = l,$$

and consequently beginning with the last

$$l - \delta = k, \; k - \delta = i, \ldots\ldots c - \delta = b, \; b - \delta = a;$$

by adding the corresponding equations, we shall perceive at once, that

$$a+l=b+k=c+i, \text{ \&c.}$$

and consequently that

$$2S=n(a+l);$$

whence it follows

$$S=\frac{n(a+l)}{2}.$$

Applying this formula to the progression

$$\div 3 . 5 . 7 . 9 \text{ \&c.}$$

we find for the sum of the first eight terms

$$\frac{(3+17)\,8}{2}=80.$$

230. The equation

$$l=a+(n-1)\,\delta,$$

together with

$$S=\frac{(a+l)\,n}{2},$$

furnishes us with the means of finding any two of the five quantities a, δ, n, l and S, when the other three are known; I shall not stop to treat of the several cases, which may be presented.

231. From proportion is derived progression by *quotients* or *geometrical* progression, which consists of a series of terms such, that the quotient arising from the division of one term by that, which precedes it, is the same, from whatever part of the series the two terms are taken. The series

$$\because\!\!\because\ 2 : 6 : 18 : 54 : 162 : \text{\&c.}$$

$$\because\!\!\because\ 45 : 15 : 5 : \tfrac{5}{3} : \tfrac{5}{9} : \text{\&c.}$$

are progressions of this kind; the quotient or *ratio* is 3 in the first, and $\frac{1}{3}$ in the other; the first is increasing, and the second decreasing. Each of these progressions forms a series of equal ratios and for this reason is written, as above.

Let

$$a, b, c, d, \ldots\ldots\ldots k, l,$$

be the terms of a progression by quotients; making $\frac{b}{a}=q$, we have by the nature of the progression,

$$q=\frac{b}{a}=\frac{c}{b}=\frac{d}{c}=\frac{e}{d}\cdots=\frac{l}{k},$$

or $\qquad b=aq,\ c=bq,\ d=cq,\ e=dq, \ldots l=kq.$

Substituting successively the value of b in the expression for c, and the value of c in the expression for d, &c. we have

$$b = a\,q,\ c = a\,q^2,\ d = a\,q^3,\ e = a\,q^4, \ldots l = a\,q^{n-1},$$

taking n to represent the place of the term l, or the number of terms considered in the proposed progression.

By means of the formula $l = a\,q^{n-1}$ we may determine any term whatever, without making use of the several intermediate ones. The tenth term of the progression

$$\div\!\!\div\ 2 : 6 : 18 : \&c.$$

for example, is equal to $2 \times 3^9 = 39366$.

232. We may also find the sum of any number of terms we please of the progression

$$:: a : b : c : d, \&c.$$

by adding together the equations

$$b = a\,q,\ c = b\,q,\ d = c\,q,\ e = d\,q, \ldots l = k\,q;$$

for the result will be

$$b + c + d + e \ldots + l = (a + b + c + d \ldots + k)\,q;$$

and representing by S the sum sought, we have

$$b + c + d + e \ldots + l = S - a$$
$$a + b + c + d \ldots + k = S - l,$$

whence $$S - a = q\,(S - l),$$

and consequently

$$S = \frac{q\,l - a}{q - 1}\dagger.$$

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† The truth of this result may be rendered very evident, independently of analysis. If it were required, for example, to find the sum of the progression

$$\div\!\!\div\ 2 : 6 : 18 : 54 : 162,$$

multiplying by the ratio we have

$$\div\!\!\div\ 6 : 18 : 54 : 162 : 486.$$

The first series being subtracted from this gives $486 - 2$, equal to so many times the first series, as is denoted by the ratio minus one, that is

$$2 + 6 + 18 + 54 + 162 = \frac{3 \times 162 - 2}{3 - 1}.$$

If we multiply by the ratio q the general series

$$\div\!\!\div\ a : b : c : d : e \ldots\ldots l$$

we have $$\div\!\!\div\ aq : bq : cq : dq : eq \ldots\ldots lq.$$

Then, because $b = a\,q$, &c. the second series minus the first is $l\,q - a$, equal to so many times the first series, as is denoted by the ratio minus one.

Hence $$a + b + c + d + e \ldots\ldots + l = \frac{lq - a}{q - 1}$$

In the above example, we find for the sum of the ten first terms of the progression

$$\div\!\!\div 2 : 6 : 18 : \&c.$$

$$\frac{2 \times 3^{10} - 2}{2} = 3^{10} - 1 = 59048.$$

233. The two equations

$$l = a q^{n-1}, \quad S = \frac{q l - a}{q - 1},$$

comprehend the mutual relations, which exist among the five quantities a, q, n, l and S in a progression by quotients, and enable us to find any two of these quantities, when the other three are given.

234. If we substitute $a q^{n-1}$ in the place of l in the expression for S, we have

$$S = \frac{a(q^n - 1)}{q - 1}.$$

When q is a whole number, the quantity q^n will become greater and greater in proportion to the increased magnitude of the number n; and S may be made to exceed any quantity whatever, by assigning a proper value to n, that is, by taking a sufficient number of terms in the proposed progression. But if q is a fraction represented by $\frac{1}{m}$, we have

$$S = \frac{a\left(\frac{1}{m^n} - 1\right)}{\frac{1}{m} - 1}\dagger = \frac{a m\left(1 - \frac{1}{m^n}\right)}{m - 1} = \frac{a m - \frac{a}{m^{n-1}}}{m - 1};$$

and it is evident, that as the number n becomes greater, the term $\frac{a}{m^{n-1}}$ will become smaller, and consequently the value of S will approach nearer and nearer to the quantity $\frac{a m}{m - 1}$, from which it will differ only by

$$\frac{a}{(m - 1)\, m^{n-1}};$$

therefore, the greater the number of terms we take in the proposed progression, the more nearly will their sum approach to

† Multiplying the numerator and denominator by $-m$.

$\frac{a\,m}{m-1}$. It may even differ from $\frac{a\,m}{m-1}$ by a quantity less than any assignable quantity, without ever becoming in a rigorous sense equal to it.

The quantity $\frac{a\,m}{m-1}$, which I shall designate by L, forms, we perceive, a limit, to which the particular sums represented by S, approach nearer and nearer.

Applying what has been said to the progression

$$\div\!\!\div\; 1 : \tfrac{1}{2} : \tfrac{1}{4} : \tfrac{1}{8} : \tfrac{1}{16}, \text{ \&c.}$$

we have

$$a=1,\quad q=\frac{1}{m}=\tfrac{1}{2},$$

whence

$$m=2,\quad L=\frac{a\,m}{m-1}=2;$$

and the greater the number of terms we take in the above progression, the nearer their sum will approach to an equality with 2.

We have, in fact

$$\begin{array}{lll}
1 & =1 & =2-1 \\
1+\tfrac{1}{2} & =\tfrac{3}{2} & =2-\tfrac{1}{2} \\
1+\tfrac{1}{2}+\tfrac{1}{4} & =\tfrac{7}{4} & =2-\tfrac{1}{4} \\
1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8} & =\tfrac{15}{8} & =2-\tfrac{1}{8} \\
1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16} & =\tfrac{31}{16} & =2-\tfrac{1}{16}. \\
\text{\&c.} & &
\end{array}$$

The expression for L may be considered as the sum of the decreasing progression by quotients, continued to infinity, and it is thus, that it is usually presented; but in order to form a clear idea of it, we must represent it in a limited view.

235. We may obtain from the expression

$$S=\frac{a\,(q^n-1)}{q-1},$$

all the terms of the progression, of which it denotes the sum; for if we divide q^n-1 by $q-1$ (158), we find

$$\frac{q^n-1}{q-1}=\frac{1-q^n}{1-q}=1+q+q^2+q^3+q^4\,\ldots\ldots+q^{n-1},$$

which gives

$$S=a+a\,q+a\,q^2\,\ldots\ldots+a\,q^{n-1}.$$

We may employ the value of L for the same purpose; in this case m is to be divided by $m-1$, as follows;

$$
\begin{array}{l|l}
m & m-1 \\
\hline
-m+1 & 1+\frac{1}{m}+\frac{1}{m^2}+\frac{1}{m^3}+\&c. \\
\hline
\quad -1+\frac{1}{m} & \\
\qquad -\frac{1}{m}+\frac{1}{m^2} & \\
\qquad\quad -\frac{1}{m^2}+\frac{1}{m^3} & \\
\qquad \&c. &
\end{array}
$$

We begin by dividing, according to the usual method, by the first term, and find 1 for the quotient; we multiply this quotient by the divisor and subtract the product from the dividend; then dividing the remainder by the first term of the divisor, we obtain $\frac{1}{m}$ for the quotient, and have $\frac{1}{m}$ for a remainder; we go through the same process with this remainder as with the preceding. Pursuing this method, we soon discover the law, to which the several particular quotients are subjected, and perceive that the expression $\frac{m}{m-1}$ is equivalent to the series

$$1+\frac{1}{m}+\frac{1}{m^2}+\frac{1}{m^3}+\&c.$$

continued to infinity. Substituting for m its value $\frac{1}{q}$, and multiplying by a, we find as before

$$a+aq+aq^2+aq^3+\&c.$$

for the progression, of which L represents the limit.

236. The series

$$1+\frac{1}{m}+\frac{1}{m^2}+\frac{1}{m^3}+\&c.$$

is considered as the value of the fraction $\frac{m}{m-1}$, whenever it is *converging*, that is, when the terms, of which it is composed, become smaller and smaller the further they are removed from the first.

Indeed, if we make the division cease successively at the first, second, third . . . . . . remainder, we have

the quotients	and the remainders
1	1
$1+\frac{1}{m}$	$\frac{1}{m}$
$1+\frac{1}{m}+\frac{1}{m^2}$	$\frac{1}{m^2}$
&c.	&c.

the former of which approach the true value, exactly in proportion as the latter are diminished; and this takes place, only when m exceeds unity. In all other cases we must have regard to the remainders, which, increasing without limit, make it evident, that the quotients are departing further and further from the true value.

To render this clear, we have only to make successively $m=2$, $m=1$, $m=\frac{1}{2}$. Upon the first supposition, we have

$$\frac{m}{m-1}=2=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\text{\&c.}$$

and it has been shown (234), that the series, which constitutes the second member, approaches, in fact, nearer and nearer to 2.

The second supposition leads us to

$$\frac{m}{m-1}=\tfrac{1}{0}=1+1+1+1+1+1+1 \text{ \&c.}$$

This result, $1+1+1+1+1$, &c. continued to infinity, presents in reality an infinite quantity, as the nature of the expression $\frac{1}{0}$ implies; yet if we neglect the remainders in this example, we are led into an absurdity; for since the divisor, multiplied by the quotient, must produce the dividend, we have

$$1=(1+1+1+1+\ldots\ldots)\,0;$$

but the second member is strictly reduced to nothing, we have therefore $1=0$.

The third supposition leads to consequences not less absurd, if we neglect the remainders, and consider the series, which is obtained, as expressing the value of the fraction, from which it is derived. Making $m=\frac{1}{2}$, we find

$$\frac{m}{m-1}=-1=1+2+4+8+16+\text{\&c.}$$

which is evidently false.

There will be no contradiction of this kind, if we observe, that in the second case, the remainders

$$1,\ \frac{1}{m},\ \frac{1}{m^2},\ \frac{1}{m^3},\ \text{\&c.}$$

are each equal to 1, and that since they do not diminish, they can never be neglected, to whatever extent the series is continued. If we add therefore one of these remainders to the second member of the equation

$$1 = (1 + 1 + 1 + 1 + 1 + \ldots\ldots)\,0,$$

the equation becomes true. In the third case, the remainders

$$1, \frac{1}{m}, \frac{1}{m^2}, \frac{1}{m^3}, \text{ \&c.}$$

form the increasing progression 1, 2, 4, 8, 16, &c. and if we add to the several quotients the fractions, arising from the corresponding remainders, the exact expressions for $\frac{m}{m-1}$ will be

$$1 + \frac{1}{m-1}$$

$$1 + \frac{1}{m} + \frac{1}{m(m-1)}$$

$$1 + \frac{1}{m} + \frac{1}{m^2} + \frac{1}{m^2(m-1)}$$

&c.

each of which gives -1, when $m = \frac{1}{2}$.

If we take $m = -n$, the fraction $\frac{m}{m-1}$ becomes $\frac{n}{n+1}$; the series, which is produced by developing this fraction, assumes the form

$$1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \text{\&c.}$$

and making $n = 1$, we have

$$1 - 1 + 1 - 1 + 1 - 1 + \text{\&c.}$$

a series, which becomes alternately 1 and 0, and which consequently as often exceeds, as it falls below the true value of $\frac{n}{n+1}$, equal in this case to $\frac{1}{2}$; but as the above series is not converging, it cannot give this true value; and we must therefore take into consideration the remainder, at whatever term we stop.

If we suppose, in the preceding series, $n = 2$, we have

$$1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16}, - \text{\&c.}$$

a series, in which the particular sums 1, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$, &c. are alternately smaller and greater than the true value of $\frac{n}{n+1}$, which is

$\frac{2}{3}$, but to which they approach continually, because the proposed series is converging.

Although *diverging* series, that is, those, the terms of which go on increasing, continue to depart further and further from the true value of the expressions from which they are derived, yet considered as developments of these expressions, they may serve to show such of their properties, as do not depend on their summation.

237. If we continue any process of division in algebra, according to the method pursued above (235) with respect to the quantities m and $m-1$, the quotient will always be expressed by an infinite series composed of *simple terms*. Infinite series are also formed by extracting the roots of imperfect powers, and continuing the operation upon the several successive remainders; but they are obtained more easily by means of the formula for binomial quantities, as will be shown in the *Supplement*, where I shall treat of the more common series.

Theory of exponential quantities and of logarithms.

238. In the several questions, we have resolved thus far, the unknown quantities have not been made subjects of consideration as exponents; this will be requisite however, if we would determine the number of terms in a progression by quotients, of which the first term, the last term, and the ratio are given. In fact, we are furnished by a question of this kind with the equation

$$l = a\,q^{n-1}\ (231),$$

in which n will be the unknown quantity; abridging the expression, by making $n-1=x$, we have $l=a\,q^x$. This equation cannot be resolved by the direct methods hitherto explained; and quantities like x cannot be represented by any of the signs already employed. In order to present this subject in a more clear light, I shall go back to state according to Euler, the connexion, which exists between the several algebraic operations, and the manner, in which they give rise to new species of quantities.

239. Let a and b be two quantities, which it is required to add together; we have

$$a+b=c;$$

and in seeking a or b from this equation, we find

$$a = c - b, \quad b = c - a;$$

nce the origin of subtraction; but when this last operation can-
t be performed in the order in which it is indicated, the result
mes negative.

The repeated addition of the same quantity gives rise to multiplication; a representing the multiplier, b the multiplicand, and c the product, we have

$$a\,b = c,$$

whence we obtain

$$a = \frac{c}{b}, \quad b = \frac{c}{a};$$

and hence arises division, and fractions, in which this division terminates, when it cannot be performed without a remainder.

The repeated multiplication of a quantity by itself produces the powers of this quantity; if b present the number of times a is a factor in the power under consideration, we have

$$a^b = c.$$

This equation differs essentially from the preceding, as the quantities a and b do not both enter into it of the same form, and hence the equation cannot be resolved in the same way with respect to both. If it be required to find a, it may be obtained by simply extracting the root, and this operation gives rise to a new species of quantities, denominated irrational; but b must be determined by peculiar methods, which I shall proceed to illustrate, after having explained the leading properties of the equation $a^b = c$.

240. It is evident, that if we assign a constant value greater than unity to a, and suppose that of b to vary, as may be requisite, we may obtain successively for c all possible numbers. Making $b = 0$, we have $c = 1$; then since b increases, the corresponding values of c will exceed unity more and more, and may be rendered as great, as we please. The contrary will be the case, if we suppose b negative; the equation $a^b = c$ being then changed into $a^{-b} = c$, or $\frac{1}{a^b} = c$, the values of c will go on decreasing, and may be rendered indefinitely small. We may therefore obtain from the same equation all possible positive numbers, whether entire or fractional, upon the supposition, that a exceeds unity. The same is true, if we have $a < 1$; only the order, in which the values stand, will be reversed; but if we suppose $a = 1$, we shall always find $c = 1$, whatever value be

assigned to b; we must therefore consider the observations, which follow, as applying only to cases, in which a differs essentially from unity.

In order to express more clearly, that a has a constant value, and that the two other quantities b and c are indeterminate, I shall represent them by the letters x and y; we then have the equation $a^x = y$, in which each value of y answers to one value of x, so that either of these quantities may be determined by means of the other.

241. This fact, that all numbers may be produced by means of the powers of one, is very interesting, not only when considered in relation to algebra, but also on account of the facility, with which it enables us to abridge numerical calculations. Indeed, if we take another number y', and designate by x' the corresponding value of x, we shall have $a^{x'} = y'$, and consequently, if we multiply y by y', we have

$$y y' = a^x \times a^{x'} = a^{x+x'};$$

if we divide the same, the one by the other, we find

$$\frac{y'}{y} = \frac{a^{x'}}{a^x} = a^{x'-x}:$$

lastly, if we take the m^{th} power of y, and the n^{th} root, we have

$$y^m = (a^x)^m = a^{mx}$$

for the one, and

$$y^{\frac{1}{n}} = (a^x)^{\frac{1}{n}} = a^{\frac{x}{n}}$$

for the other.

It follows from the two first results, that knowing the exponents x and x' belonging to the numbers y and y', we may, by taking their sum, find the exponent, which answers to the product $y y'$, and by taking their difference, that which answers to the quotient $\frac{y'}{y}$. From the two last equations it is evident, that the exponent belonging to the m^{th} power of y may be obtained by simple multiplication, and that, which answers to the n^{th} root, by simple division.

Hence it is obvious, that by means of a table, in which against the several numbers y, are placed the corresponding values of x, y being given, we may find x, and the reverse; and the *multiplication of any two numbers is reduced to simple addition*, because, instead of employing these numbers in the operation, we may add the corresponding values of x, and then seeking

in the table the number, to which this sum answers, we obtain the product required. The quotient of the proposed numbers, may be found, in the same table, opposite the difference between the corresponding values of x, and therefore *division is performed by means of subtraction.*

These two examples will be sufficient to enable us to form an idea of the utility of tables of the kind here described, which have been applied to many other purposes since the time of Napier, by whom they were invented. The values of x are termed *logarithms*, and consequently *logarithms are the exponents of the powers, to which a constant number must be raised, in order that all possible numbers may be successively deduced from it.*

The constant number is called the base of the table or system of logarithms.

I shall, in future, represent the logarithm of y by $\mathrm{l}\,y$; we have then $x=\mathrm{l}\,y$, and since $y=a^x$, we are furnished with the equation $y=a^{\mathrm{l}y}$.

242. As the properties of logarithms are independent of any particular value of the number a, or of their base, we may form an infinite variety of different tables by giving to this number all possible values, except unity. Taking, for example, $a=10$, we have $y=(10)^{\mathrm{l}y}$, and we discover at once that the numbers

$$1,\quad 10,\quad 100,\quad 1000,\quad 10000,\quad 100000,\quad \&c.$$

which are all powers of 10, have for logarithms, the numbers

$$0,\quad 1,\quad 2,\quad 3,\quad 4,\quad 5,\quad \&c.$$

The properties mentioned in the preceding article may be verified in this series; thus if we add together the logarithms of 10 and 1000, which are 1 and 3, we perceive, that their sum 4 is found directly under 10000, which is the product of the proposed numbers.

243. The logarithms of the intermediate numbers, between 1 and 10, 10 and 100, 100 and 1000, &c. can be found only by approximation. To obtain, for example, the logarithm of 2, we must resolve the equation $(10)^x=2$, by the method given in art. 221, finding first the entire number approaching nearest to the value of x. It is obvious at once, that x is between 0 and 1, since $(10)^0=1$, $(10)^1=10$; we make therefore $x=\frac{1}{z}$, the equation then becomes $(10)^{\frac{1}{z}}=2$, or $10=2^z$; now z is found

between 3 and 4; we suppose therefore $z = 3 + \frac{1}{z'}$ and hence

$$10 = 2^{3+\frac{1}{z'}} = 2^3 \times 2^{\frac{1}{z'}} = 8 \times 2^{\frac{1}{z'}},$$

or $$2^{\frac{1}{z'}} = \tfrac{10}{8} = \tfrac{5}{4},$$
or lastly $$2 = (\tfrac{5}{4})^{z'}.$$
As the value of z' is between 3 and 4, we make

$$z' = 3 + \frac{1}{z''};$$

we have then

$$2 = (\tfrac{5}{4})^{3+\frac{1}{z''}} = (\tfrac{5}{4})^3 . (\tfrac{5}{4})^{\frac{1}{z''}},$$

whence we obtain

$$(\tfrac{5}{4})^{\frac{1}{z''}} = 2\,(\tfrac{4}{5})^3 = \tfrac{128}{125}, \text{ or } (\tfrac{128}{125})^{z''} = \tfrac{5}{4};$$

and after a few trials we discover that z'' is between 9 and 10. The operation may be continued further; but as I have exhibited this process merely to show the possibility of finding the logarithms of all numbers, I shall confine myself to the supposition of $z'' = 9$; we have then, going back through the several steps,

$$z' = \tfrac{28}{9}, \quad z = \tfrac{93}{28}, \quad x = \tfrac{28}{93}.$$

This value of x, reduced to decimals, is exact to the fourth figure, as it gives

$$x = 0{,}30107.$$

By calculations carried to a greater degree of exactness it is found, that

$$x = 0{,}3010300,$$

the decimal figures being extended to seven places.

Regarding this value of x as an exponent, we must conceive the number 10 to be raised to the power denoted by the number 3010300, and the root of the result to be taken for the degree denoted by 10000000; we thus arrive at a number approaching very nearly to 2; that is $(10)^{\frac{3010300}{10000000}} = 2$, very nearly; the first member is a little greater than 2; but $(10)^{\frac{3010299}{10000000}}$ is smaller.*

* The method explained in this article becomes impracticable, when the numbers, the logarithms of which are required, are large; another method however, which may be very useful, is given by Long, an English geometer, in the *Philosophical Transactions* for the year 1724, No. 389.

244. By multiplying the logarithm of 2 successively by 2, 3, 4, &c. we obtain logarithms of the numbers 4, 8, 16, &c. which are the 2[d], 3[d], 4[th], &c. powers of 2.

By adding to the logarithm of 2 the logarithms of 10, 100, 1000, &c. we obtain those of 20, 200, 2000, &c. it is evident therefore, that if we have the logarithms of the former numbers, we may find the logarithms of all numbers composed of them, which latter can be only powers or products of the former. The number 210, for example, being equal to

$$2 \times 3 \times 5 \times 7,$$

its logarithm is equal to

$$l2 + l3 + l5 + l7,$$

and since $5 = \frac{10}{2}$, we have

$$l5 = l10 - l2.$$

As the process for determining x in the equation $(10)^x = y$ is very laborious, we may, reversing the order, furnish ourselves with the several expressions for x, then forming a table of the values of y corresponding to those of x, we shall afterwards, as will be perceived, be able, in any particular case, to determine x by means of y.

We take first for x the values comprehended between 0,1 and 0,9; we have then only to determine the value of y, which answers to $x = 0{,}1$, or $(10)^{\frac{1}{10}}$, because the several other values of y, namely

$$(10)^{\frac{2}{10}}, \quad (10)^{\frac{3}{10}}, \text{ \&c.}$$

are the 2[d], 3[d], &c. powers of the first.

By extracting the square root, we discover at once, that

$$(10)^{\frac{1}{2}} \quad \text{or} \quad (10)^{\frac{5}{10}} = 3{,}162277660;$$

then taking the fifth root of this result, we have

$$(10)^{\frac{1}{10}} = 1{,}258925412.$$

By a similar process, we deduce from

$$(10)^{\frac{1}{10}} = 1{,}258925412$$

the value of

$$\sqrt{(10)^{\frac{1}{10}}} = (10)^{\frac{1}{20}} = (10)^{\frac{5}{100}} = 1{,}122018454;$$

then taking the fifth root, we have

$$(10)^{\frac{1}{100}} = 1{,}023292992;$$

and raising the result to the 2[d], 3[d] . . . . . 9[th] powers, we obtain the values of y, corresponding to those of x comprehended between 0,01 and 0,09.

It will be readily seen, that by this method, we may also find the

245. Logarithms, which are always expressed by decimals, composed of two parts, namely, the units placed on the left the comma, and the decimal figures found on the right. The

---

ues of y for those of x between 0,001 and 0,009, between 0,0001 l 0,0009; thus we shall be furnished with the following table.

Log.	Nat. Num.	Log.	Nat. Num.
0,9	7,943282347	0,00009	1,000207254
8	6,309573445	8	1,000184224
7	5,011872336	7	1,000161194
6	3,981071706	6	1,000138165
5	3,162277660	5	1,000115136
4	2,511886432	4	1,000092106
3	1,995262315	3	1,000069080
2	1,584893193	2	1,000046053
1	1,258925412	1	1,000023026
0,09	1,230268771	0,000009	1,000020724
8	1,202264435	8	1,000018421
7	1,174897555	7	1,000016118
6	1,148153621	6	1,000013816
5	1,122018454	5	1,000011518
4	1,096478196	4	1,000009210
3	1,071519305	3	1,000006908
2	1,047128548	2	1,000004605
1	1,023292992	1	1,000002302
0,009	1,020939484	0,0000009	1,000002072
8	1,018591388	8	1,000001842
7	1,016248694	7	1,000001611
6	1,013911386	6	1,000001381
5	1,011579454	5	1,000001151
4	1,009252886	4	1,000000921
3	1,006931669	3	1,000000690
2	1,004615794	2	1,000000460
1	1,002305238	1	1,000000230
0,0009	1,002074475	0,00000009	1,000000207
8	1,001843766	8	1,000000184
7	1,001613109	7	1,000000161
6	1,001382506	6	1,000000138
5	1,001151956	5	1,000000115
4	1,000921459	4	1,000000092
3	1,000691015	3	1,000000069
2	1,000460623	2	1,000000046
1	1,000230285	1	1,000000023

y means of this table, we may find the logarithm of any number tever, by dividing it by 10 a sufficient number of times. To obtain, example, the logarithm of 2549, we first divide this number by

first of these is called the *characteristic*, because in the logarithms under consideration, which are adapted to the supposition of $a = 10$, and which are called *common logarithms*, this part shows,

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$(10)^3$ or 1000, which is the greatest power of 10 it contains; we have then

$$2549 = (10)^3 \times 2{,}549;$$

we then seek in the table the power of 10 immediately below 2,549, and find

$$(10)^{0,4} = 2{,}511886432;$$

dividing 2,549 by this last number, we have

$$2{,}549 = (10)^{0,4} \times 1{,}014775177.$$

Again seeking in the table the power of 10 immediately below 1,014775177 we find

$$(10)^{0,006} = 1{,}013911386;$$

then dividing the preceding quotient 1,014775177 by this number, we obtain a third quotient 1,000851742.

This process is to be continued, until we arrive at a quotient, which differs from unity only in those decimal places we propose to neglect.

If we consider, in the present case, the third quotient as equal to unity, the proposed number will be resolved into factors, which will be powers of 10, for we shall have

$$2549 = (10)^3 \times (10)^{0,4} \times (10)^{0,006} = (10)^{3,406},$$

from which it is evident, that 3,406 is the logarithm of the number 2549. By extending the divisions to 7 in number, this logarithm will be found to be 3,406369.

The same table enables us with still more ease to find a number by means of its logarithm, as in the following example.

Let 2,547 be the given logarithm; the number sought will be

$$(10)^{2,547} = (10)^2 \times (10)^{0,5} \times (10)^{0,04} \times (10)^{0,007};$$

it will therefore be equal to the product of the numbers

$$\begin{aligned}(10)^2 &= 100\\ (10)^{0,5} &= 3{,}162277660\\ (10)^{0,04} &= 1{,}096478196\\ (10)^{0,007} &= 1{,}016248694\end{aligned}$$

taken from the table; and will consequently be

$$2{,}547 = 1.352{,}357.$$

A table of the same kind with the above, but much more extended, has been published in England, by Dodson, the object of which is to furnish the means of finding the number answering to a given logarithm.

to what order of units the number corresponding to the logarithm belongs. The several logarithms of the numbers between 1 and 10, as they are between 0 and 1, have necessarily 0 for their characteristic; those of the numbers between 10 and 100 have 1 for their characteristic; those of the numbers between 100 and 1000 have 2; in general, the characteristic of a logarithm contains as many units, as the proposed number has figures, minus one.

246. It is important also to remark, that the decimal part of the logarithms of numbers, which are decuple the one of the other, is the same; for example,

the logarithm of	54360	is	4,7352794,
	5436		3,7352794,
	543,6		2,7352794,
	54,36		1,7352794,
	5,436		0,7352794;

for as each of these numbers is the quotient of that, which precedes it, divided by 10, the logarithm of the one is found by taking an unit from the characteristic of that of the other (241,242).

247. According to what has been said in art. 240, the logarithms of fractional numbers are, upon our present hypothesis, negative; and we may easily deduce them from those of entire numbers, if we observe that a fraction represents the quotient arising from the division of the numerator by the denominator. When the numerator is less than the denominator, its logarithm is also less, than that of the denominator, and consequently if we subtract the latter from the former, the result will be negative.

In order to obtain the logarithm of the fraction $\frac{1}{2}$, for example, we subtract from 0, which denotes the logarithms of 1, the fraction 0,3010300, which represents that of 2; the result is

$$-0{,}3010300.$$

If we subtract from 0 the number 1,3010300, which is the logarithm of 20, we have the logarithm of $\frac{1}{20}$, equal to

$$-1{,}3010300.$$

The logarithm of 3 being 0,4771213, that of $\frac{2}{3}$ will be

$$0{,}3010300 - 0{,}4771213 = -0{,}1760913.$$

248. It is evident from the manner, in which the logarithms of fractions are obtained, that, considered independently of their signs, they belong (241) to the quotients, arising from the division of the denominator by the numerator, and consequently an-

swer to the number, by which it is necessary to divide unity in order to obtain the proposed fraction. Indeed, $\frac{2}{3}$, for example, may be exhibited under the form $\frac{1}{\frac{3}{2}}$, and $l\,\frac{3}{2} = l3 - l2 = 0{,}1760913$.

It would be inconvenient in order to find the value of a fraction, to which, a given negative logarithm belongs, to employ the number to which the same logarithm answers when positive, since it would be necessary to divide unity by this number; but if we subtract this logarithm from 1, 2, 3, &c. units, the remainder will be the logarithm of a number, which expresses the fraction sought, when reduced to decimals, since this subtraction answers to the division of the numbers 10, 100, 1000, &c. by the number to which the proposed logarithm belongs.

Let there be, for example, $-0{,}3010300$; if without regarding the sign, we take this logarithm from 1, or 1,0000000, the remainder 0, 6989700, being the logarithm of 5, shows, that the fraction sought is equal to 0,5, since we supposed unity to be composed of 10 parts.

If in seeking the logarithm of a fraction, we conceive unity to be made up of 10, or 100, or 1000, &c. parts, or which amounts to the same thing, if we augment the characteristic of the logarithm of the numerator by a number of units sufficient to enable us to subtract that of the denominator from it, we obtain in this way a positive logarithm, which may be employed in the place of that indicated above.

In order to introduce uniformity into our calculations, we most frequently augment the characteristic of the logarithm of the numerator by 10 units. If we do this with respect to the fraction $\frac{2}{3}$, for example, we have

$$10{,}3010300 - 0{,}4771213 = 9{,}8239087.$$

It will be readily seen, that this logarithm exceeds the negative logarithm $-0{,}1760913$ by 10 units, and that consequently, whenever we add it to others, we introduce 10 units too much into the result; but the subtraction of these ten units is easily performed and by performing it we effect at the same time the subtraction of 0,1760913. Let N be the number, to which we add the positive logarithm 9,8239087; the result of the operation will be represented by

$$N + 10 - 0{,}1760913;$$

and if we subtract 10, we have simply

$$N - 0,1760913.$$

According to the preceding observations, we cause addition to take the place of subtraction, by employing, instead of the number to be subtracted, its *arithmetical complement*, that is, what remains, when this number is subtracted from one of the numbers 10, 100, 1000, &c. a result, which is obtained by taking the units of the proposed number from 10 and the several other figures from 9. We add this complement to the number, from which the proposed logarithm is to be subtracted, and from the sum subtract an unit of the same order as the complement.

It is evident, that if the complement is repeated several times, we must subtract after the addition, as many units of the same order with the complement, as there are in the number, by which it is multiplied; and for the same reason, if several complements are employed, we must subtract for each an unit of the same order, or as many units as there are complements, if they are all of the same order.

Sometimes this subtraction cannot be effected; in this case, the result is the arithmetical complement of the logarithm of a fraction, and answers in the tables to the expression of this fraction reduced to decimals. If 10 units remain to be taken from the characteristic, as is most frequently the case, the result is the same as if we had multiplied by 10000000000 the numerator of the fraction sought, in order to render it divisible by the denominator; the characteristic of the logarithm of the quotient shows the highest order of the units contained in this quotient, considered with reference to those of the dividend. In 9,8239087, the characteristic 9 shows, that the quotient must have one figure less than the number, by which we have multiplied unity; and consequently if we separate 10 figures for decimals, the first significant figure on the left will be tens; and we shall find only hundredths, thousands, &c. for the numbers the arithmetical complements of which have 8, 7, &c. for their characteristics.

249. What has been said respecting the *system* of logarithms, in which $a = 10$, brings into view the general principles necessary for understanding the nature of the tables; for more particular information the learner is referred to the tables themselves, which usually contain the requisite instruction relating to their arrangement and the method of using them. I will merely

mention the tables of Callet, stereotype edition, and those of Borda, as very complete and very convenient.

250. If we have the logarithm of a number y for a particular value of a, or for a particular base, it is easy to obtain the logarithm of the same number in any other system. If we have $a^x = y$; for another base A, we have $A^X = y$, X being different from x; hence we deduce $A^X = a^x$. Taking the logarithms according to the system the base of which is a, we have

$$\mathrm{l}\,A^X = \mathrm{l}\,a^x;$$

now $\mathrm{l}\,a^x = x$ by hypothesis, and $\mathrm{l}\,A^X = X\,\mathrm{l}\,A$ (241); therefore $X\,\mathrm{l}\,A = x$, or $X = \frac{x}{\mathrm{l}\,A}$; but if we consider A as a base, X will be the logarithm of y in the system founded on this base; if therefore we designate this last by $\mathrm{L}\,y$, in order to distinguish it from the other, we have

$$\mathrm{L}\,y = \frac{\mathrm{l}\,y}{\mathrm{l}\,A},$$

and *we find the logarithm of* y *in the second system, by dividing its logarithm taken in the first by the logarithm of the base of the second system.*

The preceding equation gives also $\frac{\mathrm{l}\,y}{\mathrm{L}\,y} = \mathrm{l}\,A$; from which it is evident, that whatever be the number y, there is between the logarithms $\mathrm{l}\,y$ and $\mathrm{L}\,y$, a ratio invariably represented by $\mathrm{l}\,A$.

251. In every system the logarithm of 1 is always 0, since whatever be the value of a we have always $a^0 = 1$. As logarithms may go on increasing indefinitely, they are said to become infinite at the same time with the corresponding numbers; and as, when y is a fractional number, we have $y = \frac{1}{a^x} = a^{-x}$, it is evident, that in proportion as y becomes smaller, x in its negative state becomes greater, but we can never assign for x a number, which shall render y strictly nothing. In this sense it is said, that *the logarithm of zero is equal to an infinite negative quantity,* as we find in many tables.

252. I now proceed to give some examples of the use, which may be made of logarithms in finding the numerical value of formulas. It follows from what is said in art. 241, and from the definition of logarithms, by which we are furnished with the equation $a^{\mathrm{l}y} = y$, that

$$\mathrm{l}(AB) = \mathrm{l}\,A + \mathrm{l}\,B, \quad \mathrm{l}\left(\frac{A}{B}\right) = \mathrm{l}\,A - \mathrm{l}\,B,$$

$$\mathrm{l}\,A^m = m\,\mathrm{l}\,A, \qquad \mathrm{l}\,A^{\frac{1}{n}} = \frac{1}{n}\,\mathrm{l}\,A.$$

Applying these principles to the formula

$$\frac{A^2\sqrt{B^2 - C^2}}{C\sqrt[5]{D^3\,E\,F}},$$

which is very complicated, we find

$$\mathrm{l}(A^2\sqrt{B^2 - C^2}) = \mathrm{l}\,[A^2\sqrt{(B+C)(B-C)}] =$$
$$2\,\mathrm{l}\,A + \tfrac{1}{2}\,\mathrm{l}\,(B+C) + \tfrac{1}{2}\,\mathrm{l}\,(B-C),$$

$$\mathrm{l}(C\sqrt[5]{D^3\,E\,F}) = \mathrm{l}\,C + \tfrac{3}{5}\,\mathrm{l}\,D + \tfrac{1}{5}\,\mathrm{l}\,E + \tfrac{1}{5}\,\mathrm{l}\,F,$$

and consequently

$$\mathrm{l}\left(\frac{A^2\sqrt{B^2 - C^2}}{C\sqrt[5]{D^3\,E\,F}}\right) =$$

$$2\,\mathrm{l}\,A + \tfrac{1}{2}\,\mathrm{l}(B+C) + \tfrac{1}{2}\,\mathrm{l}\,(B-C) - l\,C - \tfrac{3}{5}\,\mathrm{l}\,D - \tfrac{1}{5}\,\mathrm{l}\,E - \tfrac{1}{5}\,\mathrm{l}\,F.$$

If we take the arithmetical complements of $\mathrm{l}\,C$, $\frac{3}{5}\,\mathrm{l}\,D$, $\frac{1}{5}\,\mathrm{l}\,E$, $\frac{1}{5}\,\mathrm{l}\,F$, designating them by C', D', E', F', instead of the preceding result, we have

$$2\,\mathrm{l}\,A + \tfrac{1}{2}\,\mathrm{l}\,(B+C) + \tfrac{1}{2}\,\mathrm{l}\,(B-C) + C' + D' + E' + F',$$

only we must observe to subtract from the sum as many units of the same order with the complements, as there are complements taken, that is 4. When we have found the logarithm of the proposed formula, the tables will show the number, to which this logarithm belongs, which will be the value sought.

253. Logarithms are of most frequent use in finding the fourth term of a proportion. It is evident, that if $a : b :: c : d$ we have

$$d = \frac{b\,c}{a}, \quad \text{whence} \quad \mathrm{l}\,d = \mathrm{l}\,b + \mathrm{l}\,c - \mathrm{l}\,a;$$

that is, *the logarithm of the fourth term sought is equal to the sum of the logarithms of the two means, diminished by the logarithm of the known extreme*, or rather *to the sum of the logarithms of the means plus the arithmetical complement of the logarithm of the known extreme.*

254. If we take the logarithms of each member of the equation $\frac{b}{a} = \frac{d}{c}$, which presents the character of a proportion, we have

$$\mathrm{l}\,b - l\,a = \mathrm{l}\,d - \mathrm{l}\,c \ (252);$$

whence it follows, that the four logarithms

$$\mathrm{l}a \,.\, \mathrm{l}b : \mathrm{l}c \,.\, \mathrm{l}d$$

form an equidifference (223).

The series of equations

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \frac{e}{d} \text{ \&c. (231)}$$

leads also to

$$\mathrm{l}b - \mathrm{l}a = \mathrm{l}c - \mathrm{l}b = \mathrm{l}d - \mathrm{l}c = \mathrm{l}e - \mathrm{l}d \text{ \&c.}$$

and hence we infer, that the progression by quotients,

$$\div\!\!\div a : b : c : d : e, \text{ \&c.}$$

corresponds to the progression by differences,

$$\div \mathrm{l}a \,.\, \mathrm{l}b \,.\, \mathrm{l}c \,.\, \mathrm{l}d \,.\, \mathrm{l}e, \text{ \&c.}$$

and consequently *the logarithms of numbers in progression by quotients, form a progression by differences.*

255. If we have the equation $b^x = c$, we may easily resolve it by means of logarithms; for as $\mathrm{l}\,b^x$ is equal to $x\,\mathrm{l}\,b$, we have $x\,\mathrm{l}\,b = \mathrm{l}\,c$, and consequently $x = \frac{\mathrm{l}\,c}{\mathrm{l}\,b}$. The equation $b^{c^x} = d$ may be resolved in the same manner; making $c^x = u$, we have

$$b^u = d, \quad u\,\mathrm{l}\,b = \mathrm{l}\,d, \quad u = \frac{\mathrm{l}\,d}{\mathrm{l}\,b}, \quad \text{or} \quad c^x = \frac{\mathrm{l}\,d}{\mathrm{l}\,b};$$

again taking the logarithms, we find

$$x\,\mathrm{l}\,c = \mathrm{l}\left(\frac{\mathrm{l}\,d}{\mathrm{l}\,b}\right) = \mathrm{ll}\,d - \mathrm{ll}\,b \quad \text{and} \quad x = \frac{\mathrm{ll}\,d - \mathrm{ll}\,b}{\mathrm{l}\,c}.$$

In this last expression, $\mathrm{ll}\,b$ represents the logarithm of the logarithm of b, and is found by considering this logarithm as a number. The quantities b^x, b^{c^x}, and all which are derived from them, are called *exponential quantities.*

Questions relating to the interest of money.

256. The principles of progression by quotients and of logarithms will be found to occur in the calculations relating to interest. To understand what I have to offer on this subject it must be recollected, that the income derived from a sum of money employed in trade or in executing some productive work will be in proportion to the frequency with which it is exchanged in either case. Hence it follows, that he, who borrows a sum of money for any purpose, ought upon returning this money at the

iration of a given time to allow the lender a premium equiv- ɪt to the profits, which he might have received, if he had ployed it himself. Such is the view in which the subject of ɔrest presents itself. In order to determine the interest of r sum, we compare this sum with 100 dollars taken as unity, ring fixed the premium, which ought to be allowed for this t at the end of a particular term, one year for example. I ll not here consider those things, which in the different kinds speculation, occasion the rise and fall of interest; this belongs the elements of political and commercial arithmetic, which uld be preceded by some account of the doctrine of chances. r object in what follows is simply to resolve certain questions, ich refer themselves to progression by quotients.

To present the subject in a general point of view, I shall sup- e the annual premium, allowed for a sum 1, to be represented r, r being a fraction; it is evident, that the interest of a sum , for the same time, will be $100r$, that of any sum whatever ill be denoted by $a\,r$; if we designate this last by α, we have

$$\alpha = ar.$$

means of this formula, it is easy to find the interest of any ı whatever, when that of 100 or of any other sum, for a known e, is given; questions of this kind belong to what is called *ple interest.*

:57. But if the lender instead of receiving annually the inter- of his money, leaves it in the hands of the borrower to ac- nulate, together with the original sum, during the following ır, the value of the whole at the end of this year may be nd in the following manner. The original sum being a, if add to it the interest $a\,r$, it becomes at the end of the first ɪr

$$a + a\,r = a\,(1 + r).$$

w if we make

$$a\,(1 + r) = a',$$

interest of the sum a' for one year being $a'r$, that of the sum $1 + r)$ will be, for a second year, $a\,r\,(1 + r)$; and as, at the end the first year, the principal a, augmented by the interest, be- mes $a\ (1 + r)$, the principal a' amounts, at the end of the cond year to,

$$a'\,(1 + r) = a\,(1 + r)^2 = a''.$$

If the lender does not now withdraw the sum a'', but leaves it to accumulate during a third year, at the end of this, it will become, according to what precedes,

$$a''(1+r) = a(1+r)^3 = a'''.$$

It will be readily perceived, that a''' will become at the end of the fourth year

$$a'''(1+r) = a(1+r)^4,$$

and so on; and that consequently the sum first lent, and the several sums due at the end of the first, second, third, fourth, &c. years, form the following progression by quotients;

$$\div\, a : a(1+r) : a(1+r)^2 : a(1+r)^3 : a(1+r)^4 : \&c.$$

of which the quotient is $1+r$, and the general term

$$a(1+r)^n = A,$$

the number n representing the number of years, during which the interest is suffered to accumulate.

If the rate of interest be 5 per cent. for example, that is, if for 100 dollars during one year 105 dollars are paid back; we have

$$100r = 5, \quad \text{or} \quad r = \tfrac{5}{100} = \tfrac{1}{20}, \quad \text{and} \quad 1+r = \tfrac{21}{20}.$$

If we would know to what the sum a amounts, when left to accumulate during 25 years, we have

$$n = 25, \quad \text{and} \quad a\left(\frac{21}{20}\right)^{25}$$

instead of the original sum. The 25th power of $\frac{21}{20}$ may be easily found by means of logarithms, since we have (252)

$$l\left(\frac{21}{20}\right)^{25} = 25\, l\, \tfrac{21}{20} = 25\,(l\,21 - l\,20) = 0{,}5297322,$$

which gives

$$\left(\frac{21}{20}\right)^{25} = 3{,}386 \quad \text{nearly}, \quad A = 3{,}386\, a\,;$$

and hence it may be readily seen, that 1000 dollars will in this way amount at compound interest to 3386 dollars, at the end of 25 years.

If the sum lent were for 100 years, we should have

$$A = a\left(\frac{21}{20}\right)^{100} = 131\, a$$

nearly; thus 1000 dollars would produce, at the end of this period, a sum of 131000 dollars nearly. These examples will be

sufficient to show with what rapidity sums accumulate by means of compound interest.

258. The equation

$$A = a(1+r)^n$$

gives rise to four questions; the first, which is to find A, when a, r and n are known, presents itself, whenever we seek the amount of the principal at the end of a number n of years. I have already given an example of this.

The second, which is to find r, when a, A and n are known, occurs whenever it is required to determine the rate of interest by means of the original sum, the whole amount that has become due, and the time during which it has been accumulating; we have in this case

$$1 + r = \sqrt[n]{\frac{A}{a}}.$$

The third, which is to find a, when A, r and n are known, the formula for which is

$$a = \frac{A}{(1+r)^n},$$

has for its object to determine the principal, which it is necessary to employ in order to be entitled, after a number n of years, to a sum A.

The fourth, which is to find n, when A, a and r are known, can be resolved only by means of logarithms (238, 252). Taking the logarithm of each member of the proposed equation, we have

$$\mathrm{l}\,A = \mathrm{l}\,a + n\,\mathrm{l}\,(1+r),$$

whence

$$n = \frac{\mathrm{l}\,A - \mathrm{l}\,a}{\mathrm{l}\,(1+r)}.$$

By means of this last equation we determine how many years the principal a must remain at interest in order to amount to a sum A.

To illustrate this by an example, I shall suppose that it is required to find the time in which the original sum will be doubled, the rate of interest being 5 per cent; we have

$$A = 2a, \quad \mathrm{l}\,A = \mathrm{l}\,a + \mathrm{l}\,2,$$

and consequently

$$n = \frac{\mathrm{l}\,2}{\mathrm{l}\,1\frac{1}{20}} = \frac{\mathrm{l}\,2}{\mathrm{l}\,21 - \mathrm{l}\,20} = \frac{0{,}3010300}{0{,}0211893} = 14{,}21$$

nearly.

259. The following question is one of the most complicated,

that we meet with relating to this subject. We suppose, that the lender during a number n of years, adds each year a new sum, to the amount of this year; it is required to find what will be the value of these several sums, together with the compound interest that may thence arise, at the expiration of the term proposed. Let a, b, c, d, ... k, be the sums added the first, second, third, fourth, &c. years; the sum a remaining in the hands of the borrower during a number n of years, amounts to

$$a(1+r)^n;$$

the sum b, which remains only $n-1$ years, becomes

$$b(1+r)^{n-1},$$

the sum c, which remains $n-2$ years only, becomes

$$c(1+r)^{n-2},$$

and so on; the last sum, k, which is employed only one year, becomes simply

$$k(1+r);$$

we have therefore

$$A = a(1+r)^n + b(1+r)^{n-1} + c(1+r)^{n-2} \ldots\ldots + k(1+r).$$

By calculating the several terms of the second member separately, we obtain the value of A.

The operation is very much simplified when

$$a = b = c = d \ldots\ldots = k,$$

for in this case we have

$$A = a(1+r)^n + a(1+r)^{n-1} + a(1+r)^{n-2} \ldots\ldots + a(1+r);$$

the second member of this equation forms a progression by quotients, of which the first term is $a(1+r)$, the last term $a(1+r)^n$, the quotient $1+r$, and the sum consequently

$$\frac{a(1+r)^{n+1} - a(1+r)}{r} \ (232);$$

we have therefore in this case

$$A = \frac{a(1+r)\left[(1+r)^n - 1\right]}{r}.$$

This equation gives rise also to four questions corresponding to those mentioned in connexion with the equation

$$A = a(1+r)^n.$$

260. By reversing the case we have been considering we may represent those annual sums, or sums due at stated intervals, called *annuities*; here the borrower discharges a debt with the interest due upon it by different payments made at regular periods. These payments, made by the borrower before the

debt in question is discharged, may be considered, as sums advanced to the lender toward the discharge of the debt, the value of which sums will depend upon the interval of time between the payment and the expiration of the annuity. Thus, if we represent each sum by a, the first payment, which will take place $n-1$ years before the expiration of the term of the annuity, referred to this time, is worth $a(1+r)^{n-1}$; the second, referred to the same epoch, is worth only $a(1+r)^{n-2}$; the third, $a(1+r)^{n-3}$, and so on to the last, which amounts only to the value of a. But on the other hand the sum lent being represented by A, will be worth in the hands of the borrower, after n years, $A(1+r)^n$, which must be equal to the amount of the several payments advanced by him to the lender; we have therefore

$$A(1+r)^n = a(1+r)^{n-1} + a(1+r)^{n-2} + a(1+r)^{n-3} \ldots + a,$$

or taking the sum of the progression, which constitutes the second member

$$A(1+r)^n = \frac{a[(1+r)^n - 1]}{r},$$

an equation, in which we may take for the unknown quantity successively, the quantity A, which I shall call the *value* of the annuity, because it is the sum, which it represents, the quantity a, which is the *quota* of the annuity, the quantity r, which is the rate of interest, and lastly the quantity n, which denotes the term of the annuity. In order to find this last we must have recourse to logarithms. We first disengage $(1+r)^n$, which gives

$$(1+r)^n = \frac{a}{a - Ar},$$

then taking the logarithms, we have

$$n\,\mathrm{l}(1+r) = \mathrm{l}\,a - \mathrm{l}(a - Ar),$$

whence

$$n = \frac{\mathrm{l}\,a - \mathrm{l}(a - Ar)}{\mathrm{l}(1+r)}.$$

261. To give an instance of the application of the above formulas, I shall take the following question.

To find what sum must be paid annually to cancel in 12 *years a debt of* 100 *dolls. with the interest during that time, the rate of interest being* 5 *per cent.*

In this example the quantities given are

$$A = 100, \quad n = 12, \quad r = \frac{1}{20},$$

and the annuity a is required to be found; resolving the equation

$$A(1+r)^n = \frac{a[(1+r)^n-1]}{r}$$

with reference to the letter a, we have

$$a = \frac{Ar(1+r)^n}{(1+r)^n-1}.$$

The values of the letters A, r and n, are to be substituted in this expression; and it will be found most convenient in the first place to calculate, by the help of logarithms, the quantity $(1+r)^n$, which becomes $(\frac{21}{20})^{12}$; and

$$(\tfrac{21}{20})^{12} = 1{,}79586.$$

By means of this value we obtain

$$a = \frac{100 \cdot \frac{1}{20} \cdot 1{,}79586}{1{,}79586-1} = \frac{5 \cdot 1{,}79586}{0{,}79586};$$

and determining the values of this last expression either directly or by means of logarithms, we find

$$a = 11{,}2826;$$

an annuity of 11,28 dolls. therefore is necessary to cancel in 12 years a debt of 100 dolls. the rate of interest being 5 per cent.

262. I am prevented from entering into further details on this subject by the limits I have prescribed myself in this treatise; I will merely add therefore, that in order to compare the values of different sums, as they concern the person, who pays or receives them, they must be reduced to the same epoch, that is we must find what they would amount to when referred to the same date. A banker, for instance, owes a sum a payable in n years; as an equivalent he gives a note, the nominal value of which is represented by b, and which is payable in p years, the first sum, at the time the note is given, is worth only $\frac{a}{(1+r)^n}$, because it must be considered as the original value of a principal, which amounts to a at the expiration of n years; the sum b, for the same reason, is worth at the time the note is given $\frac{b}{(a+r)^p}$; the difference

$$\frac{a}{(1+r)^n} - \frac{b}{(1+r)^p}$$

represents therefore, according as it is positive or negative, what the banker ought to give or receive by way of balance; if this

balance is not to be paid until after a number of years denoted by q, c representing its value at the time the exchange is made, it will amount at the expiration of this term, to

$$c(1+r)^q;$$

so that it will be equivalent to

$$\left(\frac{a}{(1+r)^n}-\frac{b}{(1+r)^p}\right)(1+r)^q=a(1+r)^{q-n}-b(1+r)^{q-p}.$$

The several sums $a, b \ldots\ldots k$, in art. 259, were reduced to the time of the payment of the sum A, and in art. 260, each of the payments, as well as the sum A, was referred to the time, when the annuity was to cease.

NOTE.

(Page 81.)

In articles 66 and 75 I have interpreted the negative solutions by examination of the equation, which they immediately verify, as I done before, and this method appeared to me always exact, as object is merely to show, that these solutions have a rational se, since they resolve questions analagous to the one proposed; there are often several ways of forming these questions, and the wing, which was communicated to me by M. Français, a distinshed geometer, Professor at the School of Artillery of Mayence, emed to me more simple, than that given in these Elements.

"He thinks, that we ought to leave out of the enunciation of the estion of art. 65 the idea of the departure of the couriers, and to ppose them to have been travelling from an indefinite time; the estion then would be stated thus. *Two couriers travel the same ute in the same direction* C′ A B C (page 72); *after they have proeded, each a certain time, one finds himself in* A *at the instant that e other is in* B; *their distance and rate of going are known; it is ked, at what point of the route they will encounter each other.*"

This enunciation leads to the same equation, as that of art. 65; at "the continuity of the motion being once established, the negative lution admits of an explanation without the necessity of changing e direction of one of the couriers. Indeed, since their motion does t commence at the points A and B, but both, before arriving at ese points, are supposed to have been going in the same manner for indefinite time from C' toward B, it is easy to conceive, that the urier, who at this point is in advance of the one at A, who travels ower, must at a certain time have been behind him and have overken him before his arrival at the point A. The sign — then indiates (as in the application of Algebra to Geometry) that the distance R' is to be taken in a direction opposite to $A\,R$, which is regarded positive. The change, to be made in the enunciation, to render

the negative solution positive, is reduced to supposing, that the tw couriers must have come together before their arrival at the point A instead of its taking place afterward."

Indeed, when we place the point *R'* between *A* and *C* instead of putting it between *A* and *B*, we find $AB = BR' - AR'$, whence results the equation $y - x = a$, instead of $x - y = a$, which we first obtained; and there is no need of changing the sign of c, the second equation remaining $\frac{x}{b} = \frac{y}{c}$.

M. Français applies not less happily these considerations to the case of art. 75, by substituting, for the couriers, moveable bodies subjected to a continued motion commencing from an indefinite time. He enunciates the problem thus. "*Two moveable bodies are carried uniformly in a straight line* C B (page 80) *one in the direction* B C *and the other in the direction* C B *with given velocities; that, which is carried in the first direction, is found in* B, *a known number of hours before the other has arrived at* A; *it is asked at what point of the indefinite straight line* B C *their meeting takes place?*

The solution $x = -48^{\text{mls.}}$ implies, that the two moveable bodies met at the point *R*, before that, which is carried from *C* towards *B*, had reached the point *A*, and that the second, which moves from *B* towards *C*, was at the point *C*, where he is found when the other is at the point *A*."

The position assigned to the point *R*, verifies itself by observing, that there results from it $AC = BC - AB = cd - a$, instead of $a + cd$, as first obtained (page 80,) and consequently $\frac{x}{b} = \frac{cd - a - x}{c}$, an equation which gives $x = 48$.

In this manner there is no change to be made in the direction of the motion; indeed there is a difference in the circumstances of the problem, and as I said before, this proves, that there are several physical questions corresponding to the same mathematical relations. But the enunciations, here given, have the advantage of not breaking the law of continuity, and this is derived from the consideration of lines, which represent, in a manner the most simple and general, the circumstances of a change of sign in magnitudes. (See the *Elementary Treatise of Trigonometry and Application of Algebra to Geometry.*)

NOTE.

(page 185.)

It may be thought that, in order to discover the roots of any equation of the fourth degree

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

it would be sufficient to compare it with the product of article 183, observing to put equal to each other the quantities by which the same power of x is multiplied; and it is in this manner that most elementary writers think to demonstrate, that *an equation of any degree whatever is the product of as many simple factors, as there are units in the exponent of its degree.* It will be seen by what follows, that the reasoning by which this is attempted to be proved, is defective. We stated the proposition with qualification in article 182, because it is necessary, in order to establish it unconditionally, to shew that an equation of whatever degree has a root, real or imaginary, which is not easily done in an elementary work, and which happily is not necessary. Some remarks relating to this subject may be found in the *Supplement.*

By forming the equations

$$\begin{aligned} -a-b-c-d &= p \\ ab+ac+ad+bc+bd+cd &= q \\ -abc-abd-acd-bcd &= r \\ abcd &= s \end{aligned}$$

in order to deduce from them the value of the letters a, b, c, d, the roots of the proposed equation, the calculation would be very complicated, if, in the determination of the unknown quantities a, b, c, d, we adopt the method of article 78; but if we multiply the first of the above equations by a^3, the second by a^2, the third by a, and add these three products to the fourth, member to member, we shall have

$$-a^4 = pa^3 + qa^2 + ra + s,$$

from which we derive, by simple transposition,

$$a^4 + pa^3 + qa^2 + ra + s = 0.$$

This equation contains only a, but it is entirely similar to the one proposed. The difficulty of obtaining a therefore is the same as that of obtaining x.

"Thus," says Castillon (Mém. de Berlin, année 1789) "it is shown in every work on algebra, that an equation, of any degree we please, is formed of several simple binomials, but it is not so evident that an equation, formed by the multiplication of several simple binomials, can have such coefficients as we please."

If instead of multiplying the three first equations in a, b, c, d, by a^3, a^2, and a respectively, we multiply them by b^3, b^2, and b, or by c^3, c^2, c, or by d^3, d^2, d, and add the products to the fourth equa- we sh e in the first

$$-b^4 = pb^3 + qb^2 + rb + s,$$

$$-c^4 = \quad {}^2 + rc + s,$$

$$- \quad rd + s;$$

it follow ed to the same equation in ase of hat b, & e quantities a, b, c, d, being ma quation, it is not to be sup- no de by a different operation from of the rs and neral, if the investigation of several own iti obliged to employ for each the same reasonings, the same operations, and the same known quantities, all these quantities will necessarily be roots of the same equation.

ERRATA IN THE ELEMENTS OF ALGEBRA.

Page 29 line 9 from the bottom, *for* '$d\,e$ instead of $e\,d$,' *read* $e\,d$ instead of $d\,e$.

29 line 8 " " *for* '$a\,b\,c\,d\,e\,f$,' *read* $a\,b\,c\,e\,d\,f$.

31 12 from the top *for* 'letters are considered only' *read* numbers are not considered.

68	19	"	*for* '$+\,a$,' *read* $+\,b$.
78	1	"	*for* 'does not differ,' *read* differs.
"	7	"	*for* '$+$' '$+$,' *read* $=$ $=$.
90	30	"	*for* '50000,' *read* 150000.
126	5	"	*for* '1590,' *read* 1490.
"	"	"	'1740,' " 1750.
129	21	"	*for* '$b^2\ c^2\ d^3$,' *read* $b^2\ c^2\ d^2$.

QUESTIONS FOR PRACTICE

IN LACROIX'S ALGEBRA.

I. *Addition, Art.* 18.

1. Add the quantities $x+yz+42-29x-yz-9$.
Ans. $33-28x$.

2. Add $147a+23b-a-b+2a$.
Ans. $148a+22b$.

3. Add $11a+11ab+11abc-11a+ab-11abc$.
Ans. $12ab$.

4. Add $43a-27c-20a+7c-61b-21a+57b+20c$.
Ans. $2a-4b$.

5. Add $a+9d+a-7c+8x-a+6d+6c-7x-14d$.
Ans. $a-c+d+x$.

6. Add $7abc+ab+5c-abc+21x+9c-27xy+8abc+10x-93z+10b+31x-2ab-c+5xy-abc+33z$.
Ans. $13abc+4ab+13c+10b+52x-22xy-60z$.

II. *Subtraction, Art.* 20.

7. From $6x-8y+3$
subtract $2x+9y-2$. *Ans.* $4x-17y+5$.

8. From $5xy-8$
subtract $-3xy+1$. *Ans.* $8xy-9$.

9. From $4xy-x+xy$
subtract $2xy+2+xy$. *Ans.* $2xy-x-2$.

10. From $5x+x-8-4b$
subtract $6x-10+4b-x$. *Ans.* $2+x-8b$.

11. From $148a+47ab-23abc+1-x$
subtract $99a-47ab-8abc-2+4x$.
Ans. $49a+94ab-15abc+3-5x$.

12. From $7b - 8c + 326xy - 43b + 111c + a$
subtract $-500b - 22a - 87xy + 7c$.
Ans. $464b + 23a + 110c + 413xy$.

III. *Multiplication, Art.* 32.

13. Multiply $12ax$ by $3a$. *Ans.* $36a^2x$.

14. Multiply $3xy - 8 + 2xyz$ by xy.
Ans. $3x^2y^2 - 8xy + 2x^2y^2z$.

15. Multiply $12x^2 - 4y^2$ by $-2x^2$.
Ans. $-24x^4 + 8x^2y^2$.

16. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$.
Ans. $x^4 - y^4$.

17. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.
Ans. $x^4 + x^2y^2 + y^4$.

18. Multiply $3x^2 - 2xy + 5$ by $x^2 + 2xy - 3$.
Ans. $3x^4 + 4x^3y + 14x^2 - 4x^2y^2 + 16xy - 15$.

19. Multiply $3x^3 + 2x^2y^2 + 3y^3$ by $2x^3 - 3x^2y^2 + 5y^3$.
Ans. $6x^6 - 5x^5y^2 + 21x^3y^3 - 6x^4y^4 + x^2y^5 + 15y^6$.

IV. *Division, Art.* 46.

20. Divide $10x^2y - 15y^2 - 5y$ by $5y$.
Ans. $2x^2 - 3y - 1$.

21. Divide $3a^2 - 15 + 6a + 3b$ by $3a$.
Ans. $a - \frac{5}{a} + 2 + \frac{b}{a}$.

22. Divide $6x^4 - 96$ by $3x - 6$.
Ans. $2x^3 + 4x^2 + 8x + 16$.

23. Divide $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by $2x - 3a$.
Ans. $24x^2 - 2ax - 35a^2$.

V. *Reduction of Fractions, Art.* 50 *and* 52.

24. What is the greatest common measure of $\frac{cx + x^2}{ca^2 + a^2x}$?
Ans. $c + x$.

25. What is the greatest common measure of $\frac{x^2-1}{xy+y}$?

Ans. $x+1$.

26. What is the greatest common divisor of $\frac{x^2-y^2}{x^4-y^4}$?

Ans. x^2-y^2.

27. Reduce $\frac{x^4-b^4}{x^5-b^2x^3}$ to its lowest terms.

Ans. x^2-b^2 gr. c. d. and $\frac{x^2+b^2}{x^3}$ lowest terms.

28. Reduce $\frac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$ to its lowest terms.

Ans. $a+x$ gr. c. d. and $\frac{5a^4+5a^3x}{a^2x+ax^2+x^3}$ lowest terms.

29. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a+\frac{2x}{a}$ to equivalent fractions having a common denominator.

Ans. $\frac{9a}{12a}$, $\frac{8ax}{12a}$ and $\frac{12a^2+24x}{12a}$.

30. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$ and $\frac{x^2+a^2}{x+a}$ to fractions having a common denominator.

Ans. $\frac{3x+3a}{6x+6a}$, $\frac{2a^2x+2a^3}{6x+6a}$ and $\frac{6x^2+6a^2}{6x+6a}$.

31. Reduce $\frac{b}{2a^2}$, $\frac{c}{2a}$ and $\frac{d}{a}$ to fractions having a common denominator.

Ans. $\frac{2a^2b}{4a^4}$, $\frac{2a^3c}{4a^4}$ and $\frac{4a^3d}{4a^4}$.

VI. *Multiplication and Division of Fractions, Art.* 51.

32. What is the product of $\frac{x}{a}$ and $\frac{x+a}{a+c}$?

Ans. $\frac{x^2+ax}{a^2+ac}$.

33. What is the product of $\frac{2x}{a}$, $\frac{3ab}{c}$ and $\frac{3ac}{2b}$?

Ans. $9ax$.

34. What is the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$?

Ans. $\frac{x^4-b^4}{b^2c+bc^2}$.

35. What is the quotient of $\frac{x}{3}$ divided by $\frac{2x}{9}$? *Ans.* $1\frac{1}{2}$.

36. What is the quotient of $\frac{x+1}{6}$ divided by $\frac{2x}{3}$?

Ans. $\frac{x+1}{4x}$.

37. What is the quotient of $\frac{x^4-b^4}{x^2-2bx+b^2}$ divided by $\frac{x^2+bx}{x-b}$?

Ans. $x+\frac{b^2}{x}$.

VII. *Addition and Subtraction of Fractions, Art. 53.*

38. Add $x+\frac{x-2}{3}$ to $3x+\frac{2x-3}{4}$.

Ans. $4x+\frac{10x-17}{12}$.

39. Add $\frac{2x}{3}$, $\frac{7x}{4}$, and $\frac{2x+1}{5}$ together.

Ans. $\frac{169x+12}{60}$ or $2x+\frac{49x}{60}+\frac{1}{5}$.

40. Add together $4x$, $\frac{7x}{9}$, and $2+\frac{x}{5}$.

Ans. $\frac{158x}{45}$ or $3x+\frac{23x}{45}$.

41. From $\frac{x+a}{b}$ subtract $\frac{c}{d}$. *Ans.* $\frac{dx+ad-bc}{bd}$.

42. From $\frac{3x}{7}$ subtract $\frac{2x}{9}$ *Ans.* $\frac{13x}{63}$.

43. From $3x+\frac{x}{b}$ subtract $x-\frac{x-a}{c}$.

Ans. $2x+\frac{cx+bx+ab}{bc}$.

VIII. *Problems in Simple Equations, Art.* 82.

44. In $5x - 15 = 2x + 6$ to find the value of x.

Ans. $x = 7$.

45. In $3y - 2 + 24 = 31$ to find y.

Ans. $y = 3$.

46. In the equations $\frac{x+2}{3} + 8y = 31$ and $\frac{y+5}{4} + 10x = 192$ to find x and y.

Ans. $x = 19$ and $y = 3$.

47. Out of a cask of wine, which had leaked away one third, 21 gallons were drawn, and then being gauged it was found to be half full: how much did it hold? *Ans.* 126 gallons.

48. What two numbers are those, whose difference is 7 and sum 33? *Ans.* 13 and 20.

49. What number is that from which if 5 be subtracted, two thirds of the remainder will be 40? *Ans.* 65.

50. At a certain election 375 persons voted, and the candidate chosen had a majority of 91 votes: how many voted for each candidate? *Ans.* 233 for one, and 142 for the other.

51. A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water: what is its whole length? *Ans.* 24 feet.

52. A man arriving at Paris, spent the first day $\frac{1}{3}$ of the money he brought with him, the second day $\frac{1}{4}$, and the third day $\frac{1}{5}$, after which he had only 26 crowns left: how much did he have on arriving at Paris? *Ans.* 120 crowns.

53. A horse said to a mule, if I give you one of my sacks we shall be equally loaded, if I take one of yours I shall have twice as much as you: how many sacks had each?

Ans. The horse 7 and the mule 5.

54. A man being asked how many crowns he had, replied, if you add together a half, a third and a quarter of what I have, the sum will exceed the number of crowns I have by one: what was the number? *Ans.* 12.

55. A privateer running at the rate of 10 miles an hour discovers a ship 18 miles off making way at the rate of 8 miles an hour: how many miles can the ship run before being overtaken?

Ans. 72 miles or 9 hours.

56. A hare is 50 leaps before a grey-hound and takes 4 leaps to the grey-hound's 3; but two of the grey-hound's leaps are as much as three of the hare's: how many leaps must the grey-hound take to catch the hare? *Ans.* 300.

57. A person being asked his age, replied, that $\frac{3}{4}$ of his age multiplied by $\frac{1}{12}$ of his age would give a product equal to his age: what was his age? *Ans.* 16.

58. A person has a lease for 99 years; and being asked how much of it was already expired, answered, that two thirds of the time past was equal to four fifths of the time to come: what was the time past? *Ans.* 54 years.

59. There is a fish whose tail weighs 9lbs., his head weighs as much as his tail and half his body, and his body weighs as much as his head and tail: what is the whole weight of the fish? *Ans.* 72lbs.

60. There is a certain number, consisting of two digits, the sum of which digits is 5; and if 9 be added to the number itself the digits will be inverted: what is the number? *Ans.* 23.

61. A person found, upon beginning the study of his profession, that $\frac{1}{7}$ of his life hitherto had passed before he commenced his education, $\frac{1}{3}$ under a private teacher, $\frac{1}{3}$ at a public school and four years at the university: what was his age? *Ans.* 21 years.

62. To find a number such, that, whether it be divided into two or three equal parts, the continued product of its parts shall be equal to the same quantity. *Ans.* $6\frac{3}{4}$.

63. A person has two horses and a saddle worth 50*l.*: now if the saddle be put on the back of the first horse, it will make his value double that of the second; but if it be put on the back of the second, it will make his value triple that of the first: what is the value of each horse?

Ans. one 30*l.* and the other 40*l.*

64. To divide the number 90 into four such parts that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient shall each equal the same quantity.

Ans. The parts are 18, 22, 10, and 40.

65. By his will a father disposed of his property as follows, namely, to his oldest son he gave 100 dollars of the property and a tenth part of the residue; to the second, 200 dollars and a tenth part of the residue; to the third, 300 dollars and a tenth part of the residue; and so on to the last, always increasing the sum first paid out by 100 dollars. It appeared that the portions of all the children were alike. Required the value of the property, the number of children, and the portion of each child.

Ans. The estate was 8100 dollars, the children 9, and the portion of each 900 dollars.

66. Of a battalion of soldiers, $\frac{3}{4}$ are on duty, $\frac{1}{4}$ sick, and $\frac{3}{5}$ of the residue absent; and there are 48 officers: what is the number of persons in the battalion? *Ans.* 800.

67. A and B have the same income; A is extravagant and contracts an annual debt amounting to $\frac{1}{7}$ of his income; but B lives upon $\frac{4}{5}$ of his; at the end of 10 years, B lends A money enough to pay off his debts, and has 160*l.* to spare: what is their income? *Ans.* 280*l.*

68. A person passed $\frac{1}{6}$ of his age in childhood, $\frac{1}{12}$ in youth, $\frac{1}{7}$ and 5 years besides in matrimony, at the end of which time he had a son, who died 4 years before his father, and reached only half his father's age: at what age did the father die?

Ans. 84.

69. A shepherd, driving a flock of sheep in time of war, meets with a company of soldiers, who plunder him of half his flock and half a sheep over; and he receives the same treatment from a second, third, and fourth company, each succeeding company plundering him of half the flock the last had left and half a sheep beside, insomuch that in the end he had only 8 sheep left: how many sheep had he in the beginning?

Ans. 143.

70. A person, fifteen years after he was married, being asked the age of himself and of his wife at the time of their marriage, replied, that he was then thrice as old as his wife, but that now he was only twice as old: what were their ages?

Ans. He was 45 and she 15.

meet each other, ... at the rate of ...
...urs, and the other at the rate of ...
...ng and how far did each travel before they met?

Ans. The time was 56 hours, the spaces travelled 84 and 7...
miles.

IX. *Formation of Powers and Extraction of Roots.*

73. What is the square root of $9x^2$? (*Art.* 122.) *Ans.* $3x$.

74. What is the square root of $\frac{9x^3y^2}{4a^3}$? *Ans.* $\frac{3xy}{2a\sqrt{a}}$.

75. What is the square root of $a^4+4a^3x+6a^2x^2+4ax$ $+x^4$? (*Art.* 124.) *Ans.* $a^2+2ax+x^2$.

76. What is the square root of $x^4-2x^3+\frac{3}{2}x^2-\frac{x}{2}+\frac{1}{16}$? *Ans.* $x^2-x+\frac{1}{4}$.

77. What is the third power of $-8x^2y^3$? (*Art.* 127.) *Ans.* $-512x^6y^9$.

78. What is the fifth root of $-32x^5y^{10}$? (*Art.* 129.) *Ans.* $-2xy^2$.

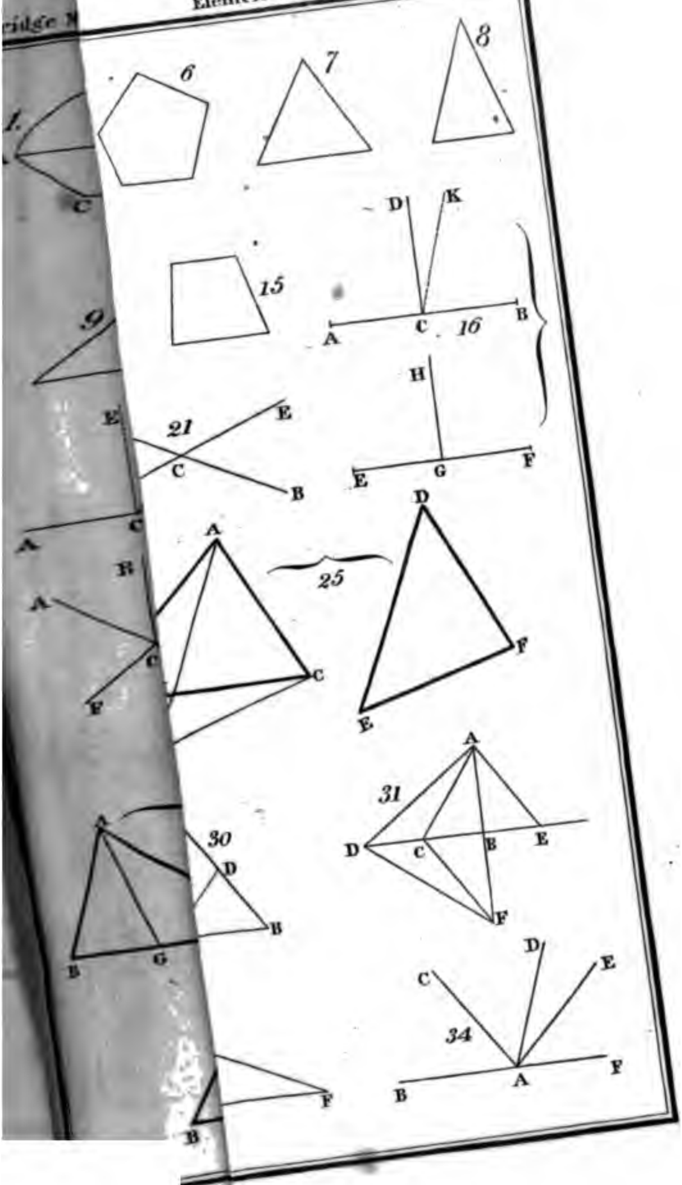
79. What is the fourth power of $x-a$? (*Art.* 141.) *Ans.* $x^4-4x^3a+6x^2a^2-4xa^3+a^4$.

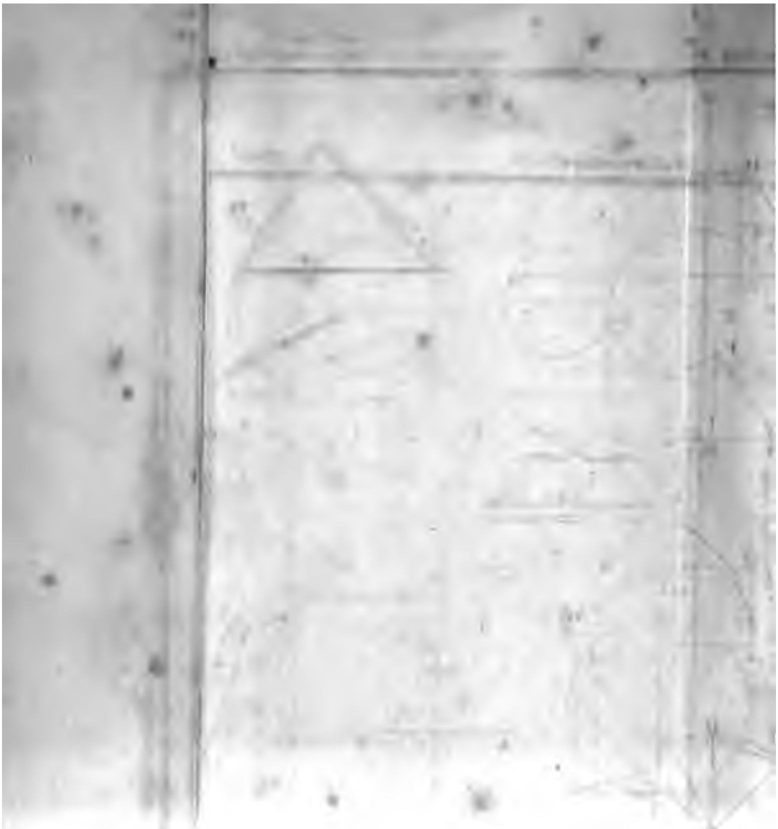
80. What is the square of $a^2+2ax+x^2$? (*Art.* 145.) *Ans.* $a^4+4a^3x+6a^2x^2+4ax^3+x^4$.

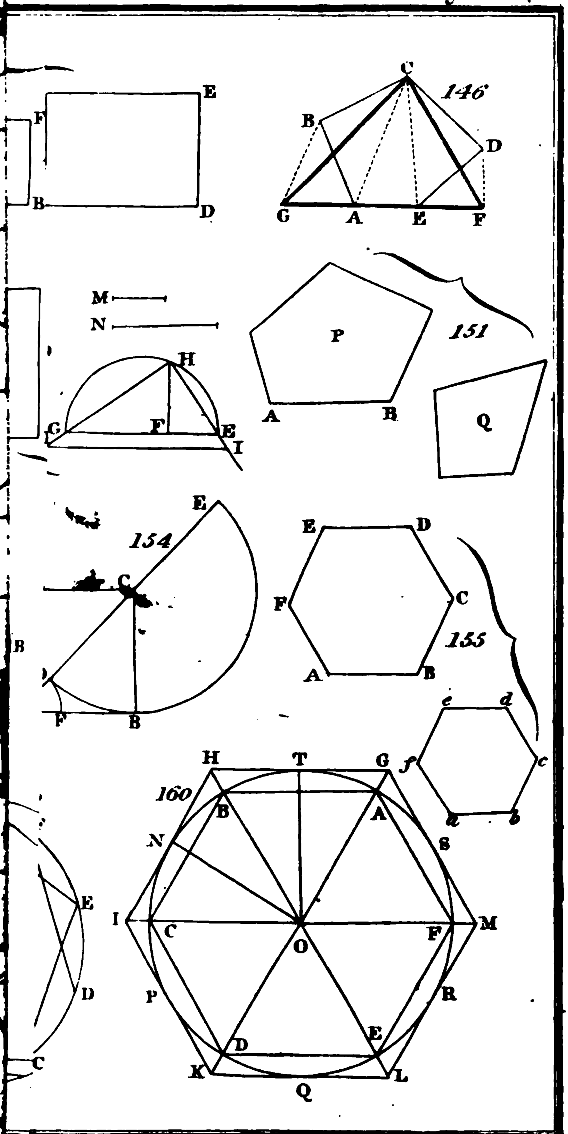
81. What is the cube root of $x^6-6x^5+15x^4-20x^3$ $+15x^2-6x+1$? (*Art.* 155.) *Ans.* x^2-2x+1.

82. What is the fifth root of $32x^5-80x^4+80x^3-40x^2$ $+10x-1$? *Ans.* $2x-1$.

Elements of Geometry. Pl. I.



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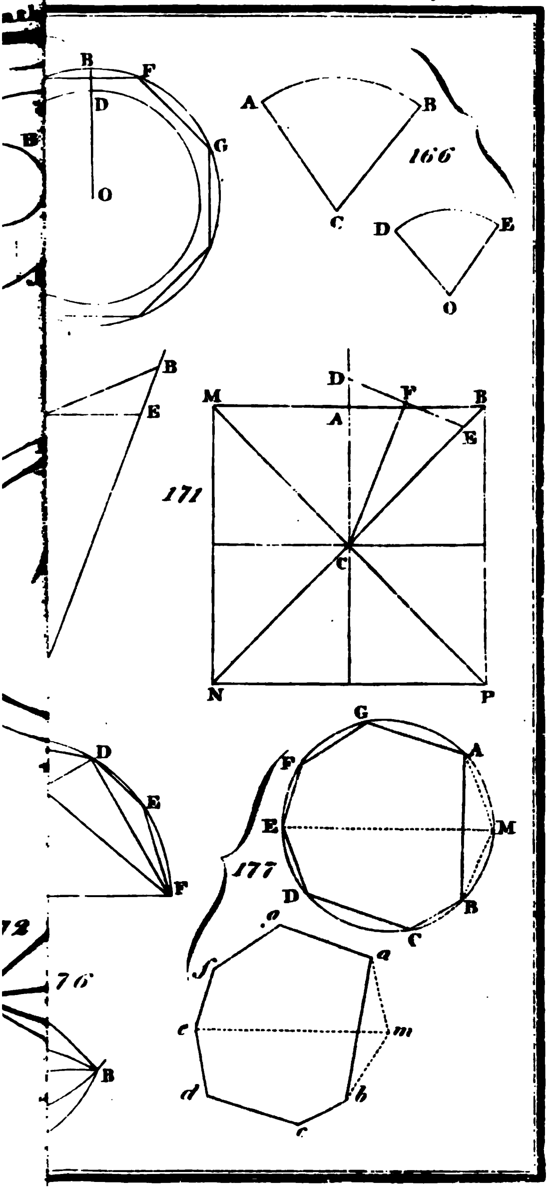
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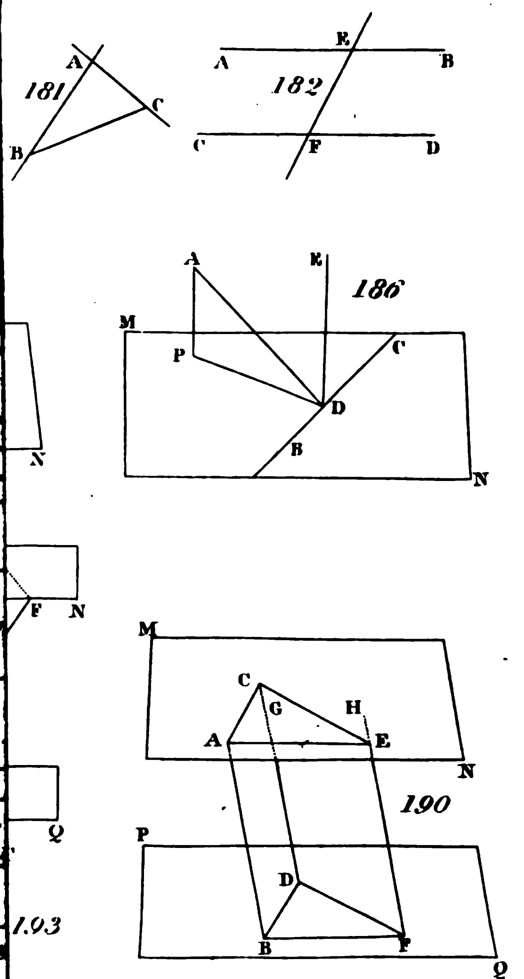
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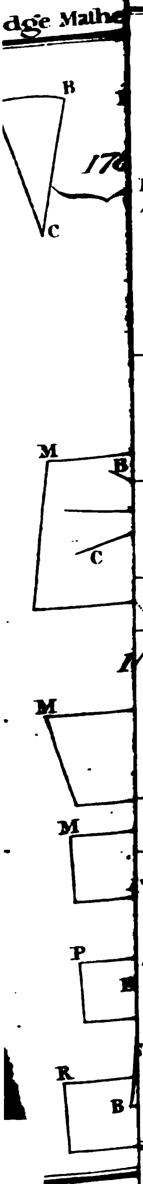

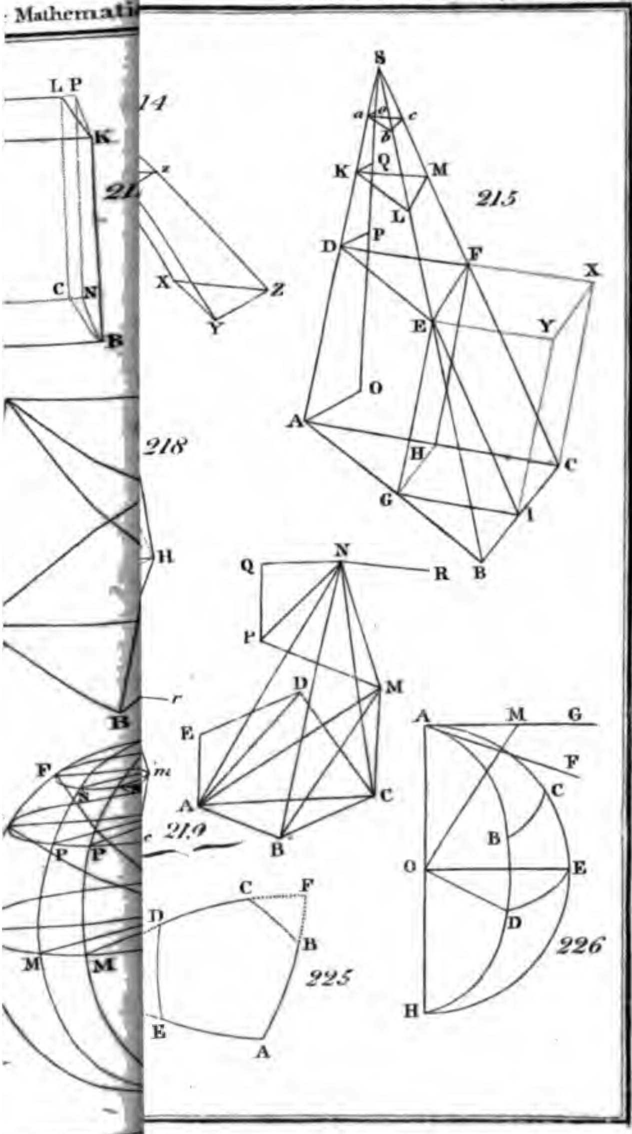
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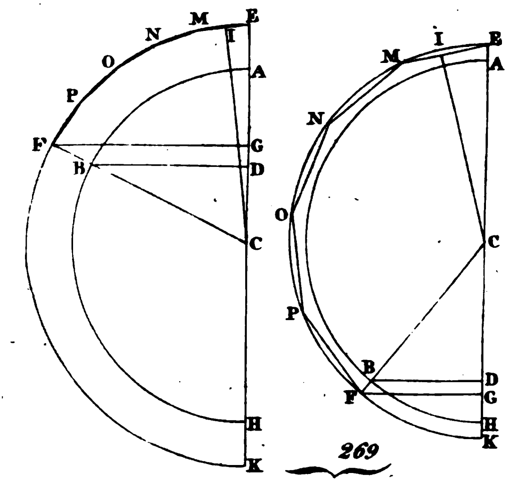

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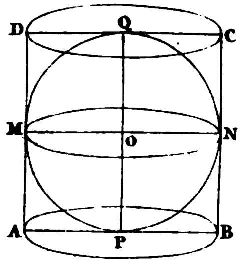

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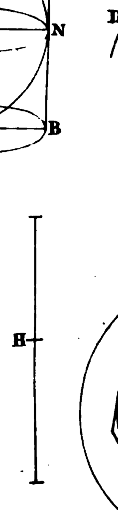

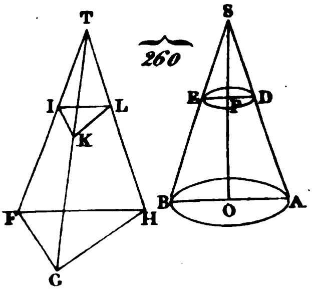

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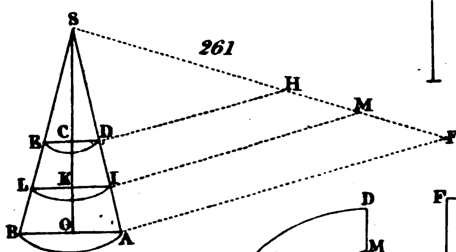

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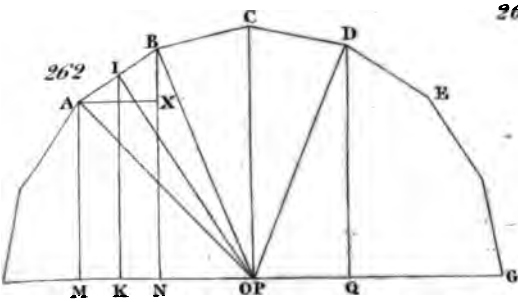

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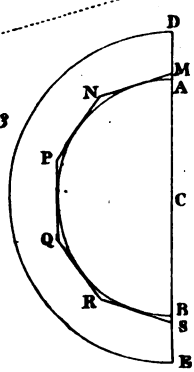

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